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Genericity in Network Dynamics

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Introduction

There could not be a better way to start any text concerned with networks than to borrow the following very first sentence from a beautiful survey over the vast field of network science:

> *"Networks are everywhere."*

NEWMAN, BARABÁSI, and WATTS [76]

The structural features of clearly distinguished agents, connections between them, and the ability to build large entities from small parts – *modularity* – do, indeed, occur abundantly. The more one investigates, the more examples can be thought of that have a natural interpretation as a network. In many scientific fields or fields of application one encounters networks in all sizes, shapes, forms, and types. In biology we find highly connected sets of neurons that make up a specific region of the brain, enormous protein interaction networks, food webs, and metabolic systems. On the other hand, we have rather sparsely connected social networks that can be vast or small in size. These have additional applications for the investigation of spreading of diseases. Furthermore, think of small systems of competing species in ecology or of power grids, electrical circuits, distribution networks, and traffic flow in engineering and economy. Another prominent field is that of computer science, where one encounters networks like the Internet (and closely related the World Wide Web and the Internet of Things), artificial neural networks in machine learning, or paths between locations in navigation systems. But also not so obvious examples can be thought of. Information or knowledge can be organized in a network to indicate connections between certain aspects (one might remember so-called Mind-Maps from school). Furthermore, everyday objects like fishing nets, weaving patterns in textiles, or spreading cracks in car windshields exhibit comparable structural features.

All these applications – and many others that have not been mentioned here – share similar structural properties as having a collection of cells or nodes (e.g. neurons, locations, humans) that are connected or communicate (e.g. cell connections, roads, communication between humans). The abundance of applications with obvious or not so obvious network structure calls for an investigation of the structural feature itself: What is a network? What properties do networks have? What implications can be deduced from network structure? As abstraction is one of its driving principles, mathematical investigation appears to be a natural choice to tackle these questions. The mathematical research of networks takes place in just as many different fields as their applications. The most obvious one, however, is graph theory. Dating back to Leonhard Euler and his famous *Seven Bridges of Königsberg* in 1736, this branch of mathematics is concerned with the investigation of mathematical objects abstractly describing network structure. A graph is used to depict such a structure in a natural manner: the agents of the network are represented by nodes and their connections by edges. But, just as the areas of applications which exhibit networks are manifold, so are the questions

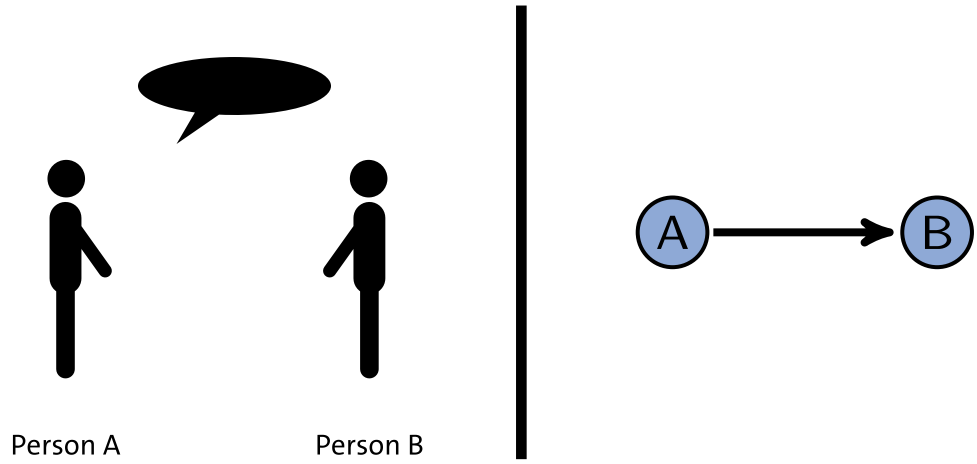


Figure 1.1: The communication of two humans depicted as a network containing two cells and one interaction.

one might ask in a given scenario: Does the network exhibit additional structure or regularity? Are there regions in the network of particular interested (e.g. clusters of all-to-all connected nodes, cells with many connections compared to other cells which are called hubs)? How do the properties of the network change, when it is varied (e.g. removal or addition of cells or connections)? What is a shortest path between two cells? How can technical networks be kept under control? Can we understand properties or behavior of the network from smaller parts of it? How is the overall behavior impacted by the network structure when the behavior of each node is influenced by that of the cells it receives inputs from? In effect, while graph theory focuses on investigating the *structural properties* of a network itself, many other fields of mathematics encounter the problem to determine *implications* of such a structure. Examples include algebra, stochastics, statisctics, control theory, and dynamical systems.

Here, we focus on the latter of these fields: *network dynamical systems*. In the investigation for example of stability of power grids, spreading of diseases, brain activity, or interactions of cells one deals with time dependent networks. Its cells are evolving agents whose behavior is influenced by that of those cells that provide an input according to the interaction structure. Oftentimes, the agents are modeled by coupled nonlinear maps or differential equations, where the structure of couplings is given by the network – we also refer to these systems as *coupled cell systems*. Other formalisms to represent more general types of dynamics are possible as well (see Chapter 2 for a brief survey). Network dynamical systems exhibit a remarkably rich spectrum of dynamical features. Interestingly, knowledge of the governing principles of the dynamics of one cell and of the nature of the interactions between two nodes is in general not sufficient to understand the dynamics of the entire network. The global structure has a major effect as well. In particular, even if the dynamics of individual cells and their interactions ought to be relatively simple and well understood, the network structure may still be the source of remarkable and unexpected dynamics. Note for example the challenges that interconnectedness of power grids pose in the prevention of blackouts. Some of the most staggering examples are *synchronization* and *pattern formation*. These terms describe collective behavior of all or some cells of the network. It is observed in social cooperation, coupled lasers, simultaneous firing of neurons, cardiac pacemaker cells, seemingly organized movement in fish schools and bird flocks, or synchronized flashing of fireflies (O'Keeffe, Hong, and Strogatz [86] and Pikovskij, Rosenblum, and Kurths [89]) and many others. If there are multiple subsets of synchronized cells, interesting patterns may arise that are sometimes referred to as *patterns of synchrony* (e.g. Golubitsky and Stewart [51, 53], Golubitsky, Stewart, and Török [59], and Stewart, Golubitsky, and Pivato [108]). Other generalizations of synchronization, most notably in networks of oscillators,

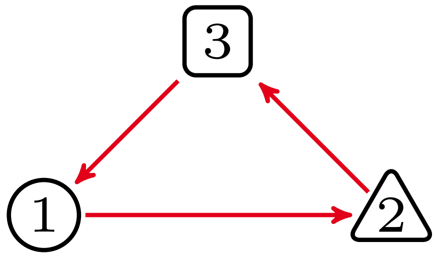


Figure 1.2: A 3-cell non-homogeneous directed circle.

are *phase-locked dynamics* (Ermentrout [33] and Glass and Mackey [41]), in which the nodes oscillate with the same frequency but out of synchrony by a fixed value, and so-called *chimeras* (Abrams and Strogatz [1] and Kuramoto and Battogtokh [70]), describing localized synchronization of frequencies.

Due to its richness in interesting features, the mathematical analysis of network dynamical systems is highly challenging. The modern theory of dynamical systems focuses on *qualitative* features of dynamics (Guckenheimer and Holmes [60]). For example synchrony and the information which cells synchronize is a qualitative property of a given network. The precise state at which these cells synchronize, however, is not. On a more technical note, qualitative dynamical properties remain under suitable changes of coordinates. The precise meaning of suitability depends on the context. For example in network dynamics one considers coordinate changes that respect the underlying network structure. The standard methodology to detect, describe, and analyze qualitative dynamical features, however, often fails to distinguish between a network dynamical system and one without such structure. For example, in order to compute suitable changes of coordinates, one has to consider concatenations of vector fields which govern the right hand side of a system of ordinary differential equations. Vector fields that respect the simple network structure in Figure 1.2 are of the form

$$F(x) = \begin{pmatrix} f_1(x_1, x_3) \\ f_2(x_2, x_1) \\ f_3(x_3, x_2) \end{pmatrix}. \tag{1.1}$$

However, concatenating F and G of this form yields a vector field

$$H(x) = \begin{pmatrix} h_1(x_1, x_3, x_2) \\ h_2(x_2, x_1, x_3) \\ h_3(x_3, x_2, x_1) \end{pmatrix} = \begin{pmatrix} f_1(g_1(x_1, x_3), g_3(x_3, x_2)) \\ f_2(g_2(x_2, x_1), g_1(x_1, x_3)) \\ f_3(g_3(x_3, x_2), g_2(x_2, x_1)) \end{pmatrix},$$

which is no longer of the form (1.1). Therefore, it is a serious challenge, to describe the coordinate changes that leave the network structure intact (Golubitsky and Stewart [56]). From a broader perspective, one of the main reasons lays in the fact that network systems behave vastly different from dynamical systems without additional structure. Characteristics, like synchrony, partial synchrony, et cetera are unheard of in general dynamical systems. On the other hand, as they are inherent to the network setting, these features are precisely what one is interested in when investigating network dynamical systems. In particular, one tries to determine what kind of dynamics is dictated by a specific network structure. Hence, even though the techniques are applicable in the network context, in order to take the underlying structure into account, they have to be adapted first.

In recent years, numerous approaches and formalisms have been developed to analyze network dynamical system while addressing the aforementioned issues. Special care has to be taken on how to encode network structure in the theory of dynamical systems, which also allows for distinction of the approaches. Furthermore, some formalisms are only suitable or most powerful in certain classes of networks so that some of the approaches complement each other. The first notable theory is the algebraically inspired *groupoid formalism* developed by Golubitsky, Stewart et al. [53, 59, 108] and in an essentially equivalent combinatorial definition by FIELD [35]. Among other things, it allows to characterize all possible configurations of synchrony in the network in terms of so-called *balanced colorings*. The introduction of the category-theoretic tool of *graph fibrations* and their implications for dynamical systems (see DEVILLE and LERMAN [24, 25, 26]) allowed for the development of a theory to interpret networks as algebraic structures – *semigroup networks* – that have a direct connection to symmetry, so-called *hidden symmetry* (NIJHOLT and RINK [77], NIJHOLT, RINK, and SANDERS [78, 79], and RINK and SANDERS [90, 91, 92]). This theory applies to the class of networks with *asymmetric inputs* – all incoming arrows into one cell are of different types – and most results focus on *homogeneous networks* – all cells are of the same type. More recently *open systems* (LERMAN [72], LERMAN and SPIVAK [73], and SCHULTZ, SPIVAK, and VASILAKOPOULOU [96] only to name a few references) and *asynchronous networks* (BICK and FIELD [19, 20, 21]) have been introduced to model networks in more complex applications especially in engineering. In Chapter 2 we provide a brief survey over some of these approaches.

Contents and contribution of the thesis

In this thesis we focus mainly on the formalism of homogeneous coupled cell systems with asymmetric inputs. Our results emerge from the investigation of *bifurcations* in the context of networks. One is interested in the determination of changes of the qualitative behavior of a network, if the dynamical system itself changes without varying the network structure, for example due to parameter dependence. Oftentimes, that means analyzing changes in the topology of the set of steady state or periodic solutions near a fully synchronous solution that changes its stability properties, so-called *synchrony breaking bifurcations*. We investigate such bifurcations that are dictated by the network structure and independent of the specific dynamical system. These are referred to as *generic bifurcations*, which is the main theme of this thesis. The research literature in the field is extensive. Investigations range from small examples over big networks, to qualitative statements for entire classes of networks: e.g. AGUIAR et al. [8], ANTONELI, DIAS, and PAIVA [14], DIAS and LAMB [27], ELMHIRST and GOLUBITSKY [32], GANDHI et al. [40], GOLUBITSKY and LAUTERBACH [42], GOLUBITSKY, PIVATO, and STEWART [45], GOLUBITSKY and POSTLETHWAITE [46], GOLUBITSKY, STEWART, and SCHAEFFER [58], KAMEI [65, 66], LEITE and GOLUBITSKY [71], NIJHOLT and RINK [77], RINK and SANDERS [90, 91], SOARES [98, 99], and STEWART and GOLUBITSKY [107], without claiming completeness of this list. The main results in this thesis are concerned with three major topics, that we briefly introduce here in the upcoming sections: *networks and symmetry*, *dimension reduction*, and so-called *feedforward networks*. The main part, on the other hand, is separated into only two main parts. We outline its organization and how the three major topics are divided afterwards.

Generalized symmetry

Symmetries arise frequently in nature and applications. In dynamical systems, symmetries impose restrictions on the governing equations leading to numerous staggering phenomena such as pattern formation or synchronization of behavior. Just as in the case of networks, the symmetries of a dynamical system provide the underlying structure that dictates this unexpected behavior. More technically speaking, they may lead to dynamically invariant subspaces, spectral degeneracies, complicated bifurcations, and many more. Research in this area – which is called *equivariant dynamics* – has been highly active in the last decades and lots of remarkable results have been established. Background and more details on equivariant dynamics can be found, for example, in CHOSSAT and LAUTERBACH [23], FIELD [36], GOLUBITSKY and SCHAEFFER [49], and GOLUBITSKY, STEWART, and SCHAEFFER [58] with no claim of this list being complete. The symmetries in question, throughout all these results, need to have an underlying structure themselves. In particular, they are required to form a *group*, most often a finite one or a compact Lie group. That is the set of symmetries needs to be closed under concatenations and, more importantly, all symmetries need to be invertible. This allows for the introduction of algebraic methods like character theory into dynamical systems.

On the other hand, the rapid development in the field of network dynamical systems has called for less restrictive classes of symmetries. The unusual phenomena exhibited in network dynamics that were mentioned before like unexpected and complex bifurcation behavior due to eigenvalues with high multiplicities and high-dimensional center subspaces resemble those in equivariant dynamics. This may come as no surprise, if the network itself exhibits symmetry. Much research, especially in the early days of network dynamics, explores these effects, as more general networks seemed out of reach (see for example ASHWIN and SWIFT [17], GOLUBITSKY, LUSS, and STROGATZ [43], and GOLUBITSKY et al. [57]). However, the observation is not limited to symmetric networks. As a matter of fact, in some of the mentioned approaches to network dynamics one encounters dynamical systems that are equivariant with respect to linear symmetries as well. These, however, often do not form a group but less restrictive structures such as a *groupoid* in the groupoid formalism and a *semigroup* or a *monoid* in semigroup networks.

In this thesis we focus almost exclusively on homogeneous networks with asymmetric inputs. Under mild additional assumptions, these can be regarded as the restriction of equivariant systems to some invariant subspace – more precisely to a subspace of partially synchronous states (see NIJHOLT, RINK, and SANDERS [78] and RINK and SANDERS [90, 91]). After recapping the formalism in Chapter 3 and some background on representation theory of monoids in Chapter 4, we thoroughly investigate generic steady state bifurcations in homogeneous coupled cell systems in the remainder of Part II. Numerous results on how to exploit symmetry to analyze the dynamics are known: Lyapunov-Schmidt reduction (RINK and SANDERS [91]), normal forms (RINK and SANDERS [92]), center manifold reduction (NIJHOLT, RINK, and SANDERS [79]), or determination of bifurcations in the extended system (NIJHOLT, RINK, and SANDERS [80]). These techniques provide a step-by-step machinery to determine generic bifurcations in the network. In particular, using Lyapunov-Schmidt and center manifold reduction, it is standard to find branching solutions (steady state or periodic) within the center manifold, which is a graph over the center subspace (GUCKENHEIMER and HOLMES [60]). This is referred to as the bifurcation occurring 'along the center subspace'.

In Chapters 5 and 6, we recapitulate our results from [97] on generic steady-state bifurcations in 1-parameter families of smooth vector fields that are equivariant with respect to the representation of a monoid and their implications for homogeneous coupled cell systems. Until this publication, one important step could not be completely clarified, namely the determination of generic center subspaces of the equivariant system. In a general 1-parameter family – without any symmetry – the kernel of the linearization is generically one-dimensional. Hence, a generic steady state bifurcation occurs along a one-dimensional subspace. In the context of group equivariant dynamics the picture is more complicated. Symmetry may dictate high multiplicities of eigenvalues leading to high-dimensional center subspaces. However, it is well-known (see for example the mentioned literature on equivariant dynamics) that generic steady state bifurcations occur along a so-called absolutely irreducible subrepresentation. This term refers to a subspace that is invariant under the symmetries and does not contain any further nontrivial invariant subspaces while fulfilling an additional algebraic condition.[1]

A similar result was only known in a special case for 1-parameter families of systems that are monoid equivariant. A monoid representation decomposes as a direct sum of subrepresentations that are indecomposable, meaning they cannot be decomposed any further. Just as in the group case they can be of three types depending on their algebraic properties. RINK and SANDERS [91] show that whenever the representation decomposes into indecomposable subrepresentations that are pairwise nonisomorphic, steady state bifurcations in 1-parameter families generically occur along an absolutely indecomposable subrepresentation. This is a generalization of the corresponding sta-

[1]The set of linear maps on this subspace, that commute with the symmetries – its *endomorphisms* – forms a division algebra which is isomorphic to the real numbers.

tement in the group symmetric context. It is already anticipated in their paper that this result holds in full generality. However, no proof is given. We were able to close this gap by proving

Theorem 1.1 (Theorem 5.11)**.** *Steady state bifurcations in* 1*-parameter families of systems that are equivariant with respect to a finite-dimensional representation of a monoid generically occur along an absolutely indecomposable subrepresentation.*

This is the generalization of the aforementioned result in the group context to monoid representations and can be applied to any monoid equivariant bifurcation problem independent of the network context.

Shortly after that publication, a further generalization to generic bifurcations in l-parameter families of monoid equivariant vector fields has been provided (see the preprint [77]). The authors address the same question from a more general point of view by determining which configurations of invariant subrepresentations generically occur as generalized kernel or as center subspace. The result of our theorem is covered as a special case. The proof, however, is a lot more involved. It is presented in a highly algebraic setting and makes use of algebraic geometry as well as noncommutative algebra – most importantly in the form of Wedderburn's structure theorem. Our theorem and in particular its proof, on the other hand, are presented from the perspective of the application to homogeneous coupled cell networks. Therefore, it does not provide the result in more generality. It may, however, serve as lighter introduction into the theory of monoid representations coming from network dynamical systems.

Networks with high-dimensional internal dynamics

A major issue in determining generic bifurcations in a given network is the computational complexity stemming from high-dimensional phase spaces. The *total phase space* – i.e. the phase space of the entire network dynamical system – has spatial components according to the *internal phase space* of each cell of the network. Hence, even if the network structure is sufficiently simple and contains only a few cells (for example the 3-cell network in Figure 1.3), the dimension of the total phase space can be large. More precisely it is the sum of the dimensions of the internal phase spaces (in the example, if the internal phase space of each cell is 5-dimensional, the total phase space is 15-dimensional). This makes the classification of bifurcations for a given network difficult and computationally costly. In order to reduce this complexity to its minimum one often restricts to the case where the internal phase space is one-dimensional. This, however, in general means a restriction of generality so that additional work is required either motivating this restriction or proving that it is not a restriction after all in a specific setting (see for example LEITE and GOLUBITSKY [71] and RINK and SANDERS [91]).

In the following example we illustrate in a very simple network an effect high-dimensional internal dynamics may have on the dynamical analysis compared to one-dimensional internal dynamics.

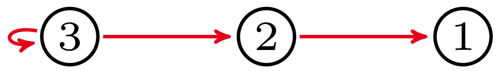


Figure 1.3: A 3-cell homogeneous feedforward chain.

Example 1.1. Consider the 3-cell homogeneous feedforward chain in Figure 1.3. Its dynamics is governed by the system of ordinary differential equations

$$\begin{aligned}\dot{v}_1 &= f(v_1, v_2)\\ \dot{v}_2 &= f(v_2, v_3)\\ \dot{v}_3 &= f(v_3, v_3),\end{aligned}$$

where $v_i \in V$ is the state variable of cell i in the internal phase space V. In order to investigate dynamical phenomena – in particular stability – one analyzes spectral properties of linearizations of the right hand side of such systems. Especially for the investigation of generic steady state bifurcations one is interested in spectral properties of a generic linear right hand side – that is, a linear admissible map –, which is of the form

$$L = \begin{pmatrix} A & B & 0 \\ 0 & A & B \\ 0 & 0 & A+B \end{pmatrix},$$

where $A, B \in \mathfrak{gl}(V)$ are generic linear maps on V. The spectrum of L is made up of the eigenvalues of A and those of $A + B$, where the eigenvalues of A occur with algebraic multiplicity 2, even though they are generically simple as eigenvalues of A. This spectral degeneracy – a double eigenvalue is unheard of in a generic linear map without any additional structure – is independent of the dimension of the underlying space V.

However, the investigation of generic steady state bifurcations also relies on information about the generalized *eigenspaces* of the linearization at a bifurcation point. Let us, for simplicity, assume that A has an eigenvalue 0. In the case $V = \mathbb{R}$ this is equivalent to the assumption $A = 0$. In that case we also have $B \neq 0$ generically. The generalized eigenspace of the eigenvalue 0 is spanned by an eigenvector and a generalized eigenvector as

$$\mathbb{E}_0 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{B} \\ 0 \end{pmatrix} \right\rangle.$$

If, on the other hand, $V = \mathbb{R}^d$ for some $d > 1$, the eigenvalue 0 of A is generically simple. Hence, there is an eigenvector $v \in V$ such that $Av = 0$ and no other (generalized) eigenvector. Furthermore, generically Bv will not be a scalar multiple of v so that we find an element $w \in V$ with

$$Aw = (\mathbb{1}_V - B)\, v.$$

Then the generalized eigenspace of L is spanned by an eigenvector with corresponding generalized eigenvector

$$\mathbb{E}_0 = \left\langle \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} w \\ v \\ 0 \end{pmatrix} \right\rangle.$$

This structure significantly differs from the one in the case $V = \mathbb{R}$. The generalized eigenvector depends not only on B but also on A. As A and B do not necessarily commute,

$$\begin{pmatrix} 0 \\ B^{-1}v \\ 0 \end{pmatrix}$$

is in general not a generalized eigenvector. Summarizing, already this simple 3-cell feedforward chain produces significantly different spectral properties when the internal dynamics is high-dimensional. $\triangle$

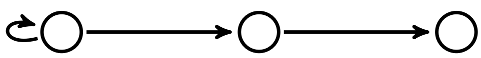
In Chapter 7 we investigate the effects of high-dimensional internal dynamics on generic steady state bifurcations in homogeneous coupled cell systems, by comparing them to networks with one-dimensional internal dynamics. We show that issues, such as the one illustrated in Example 1.1, only have a 'controllable' qualitative impact on generic steady state bifurcations in homogeneous coupled cell systems. Using representation theory and the results from Chapter 5, we show that critical eigenspaces – center subspaces or kernels – in bifurcation analysis of networks with high-dimensional internal dynamics are in some sense the same as the ones we encounter in the one-dimensional case. In Theorem 7.12, we prove that the decomposition of the monoid representation related to the network structure with high-dimensional internal dynamics is isomorphic to multiple copies of the decomposition one obtains for the one-dimensional internal dynamics case. In particular, this allows for dimension-reduction. Then we investigate implications for possible center subspaces in steady state or Hopf bifurcation analysis. Together with the knowledge about generic center subspaces in l-parameter families from Nijholt and Rink [77] this allows for statements about generic bifurcations in the high-dimensional case in comparison to the one-dimensional case. In Theorems 7.17, 7.18 and 7.22 to 7.24 we formalize these results, by proving that qualitative properties in 1-parameter bifurcations are the same. In l-parameter bifurcations, however, the situation can be more complex. Nevertheless, there is a smallest dimension of internal dynamics in which all generic bifurcations that are inherent to the network can be observed. The contents of Chapter 7 stem from the preprint [83].

We would like to mention that in the very recent preprint Gandhi et al. [40] a similar strategy to address steady state bifurcations in networks with high-dimensional internal dynamics is employed. Therein, 1- and 2-parameter bifurcations in fully inhomogeneous networks – all cells are of pairwise different types – are classified in the one-dimensional case. Then it is shown that this classification also holds true for networks with high-dimensional internal dynamics. The reason for this, however, is entirely different. As a matter of fact, the class of fully inhomogeneous networks allows to interpret a cell with d-dimensional internal dynamics as d cells of different types with one-dimensional internal dynamics that are all-to-all coupled. The classification of bifurcations in the one-dimensional case then applies directly to high-dimensional case as well.

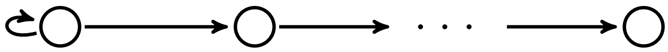
Feedforward networks

A common approach to networks is the investigation of additional structure of the network itself. A prominent example is that of *feedforward structure*. Broadly speaking a network exhibits feedforward structure if information only flows in one direction. Information one cell emits cannot become an input into that same cell, not even indirectly, i.e. there are no feedback effects. A simple network of this type can always be arranged in way that all arrows approximately point in the same direction, which provides a nice visual interpretation. The rather simple structure has the convenient effect, that it induces a natural partition of the cells: the first part receives no inputs from anywhere else in the network, the second part receives inputs only from the first and so on. This greatly simplifies the investigation of such networks both in mathematical analysis due to technical simplifications but also conceptually as it allows to study the network inductively. While feedforward structure and similar weaker properties are abundant in networks – as a matter of fact, in any network, that is not (indirectly) all-to-all coupled, we can find feedforward structure between parts of the network –, they are also a prominent feature in deep learning via artificial neural networks. Therein information of some type is passed to the so-called input layer. From there it is passed to and processed by cells in one or more so-called hidden layers until some output is generated by the so-called output layer. For more information and historical background on this (see Schmidhuber [95] and the extensive lists of references therein).

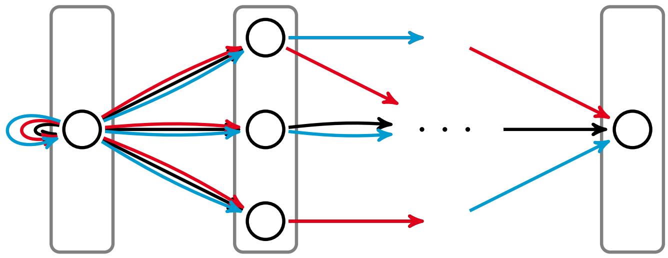
The network of feedforward type that was first considered in the network dynamical systems literature is the 3-cell homogeneous network



(see Elmhirst and Golubitsky [32], Golubitsky and Postlethwaite [46], and Golubitsky and Stewart [53]). The first cell is not influenced by any other cell, the second only by the first, and the last only by the second. We refer to this setting as a *feedforward chain*. Note that the self-arrow of the first cell, although it seems to contradict the 'no-feedback' assumption, is due to a mere convention where we allow cells to influence themselves. It was observed that dynamical systems with the underlying structure of this network exhibit surprising generic Hopf bifurcations: if a fully synchronous steady state looses stability through a pair of imaginary eigenvalues the corresponding system exhibits a Hopf branch of periodic solutions in which the first cell remains in the steady state, the amplitude of the second cell grows with rate $\sim |\lambda|^{\frac{1}{2}}$, and the amplitude of the last cell grows with rate $\sim |\lambda|^{\frac{1}{6}}$, where λ is the bifurcation parameter. In particular the growth in the last cell is much faster than expected in Hopf bifurcations. The effect is also referred to as *amplification* and is forced by the network structure. In RINK and SANDERS [90], the anomalous Hopf bifurcation result is generalized to feedforward chains of arbitrary length,



where the amplification is observed to increase the 'further down in the chain' the cell is located. Furthermore, a similar result for steady state bifurcations is proved. Since then, attempts were made to understand more general classes of feedforward networks, not restricted to chains. In NIJHOLT, RINK, and SANDERS [80], as an example, the authors introduce so-called *ring-feedforward networks* which are feedforward chains where the first cell is replaced by an oriented ring. Most recently, the amplification in steady state bifurcations (as well as other investigations) has been generalized to *layered feedforward networks*:



(see SOARES [99]). Therein the cells can be partitioned into layers such that the feedforward structure respects these layers. In particular, if we collapse each layer to one cell, we are left with a feedforward chain. Note that recently research has also extended to networks that do not exhibit a strict feedforward structure. In AGUIAR, DIAS, and FIELD [7] the authors investigate the effect of feedback loops on the synchrony patterns of weighted feedforward networks with additive input structure. Furthermore, in the preprint GANDHI et al. [40] the feedforward structure of transitive components is exploited to thoroughly investigate 1- and 2-parameter steady state bifurcations in fully inhomogeneous networks. A similar investigation is made in AGUIAR, DIAS, and SOARES [12] for one specific 1-parameter steady-state bifurcation scenario in homogeneous networks with asymmetric inputs. In this class of networks or respectively in this bifurcation problem, however, amplification is generically not possible.

Part III is entirely devoted to the investigation of a more general class of feedforward networks. Interestingly, multiple equivalent definitions are possible, the most general of which only incorporates the illustrative idea that a feedforward network should not contain any feedback (except for self-loops). This definition is independent of the class of networks (i.e. homogeneous or otherwise). The results, in particular concerning generic steady state bifurcations, however, are coined towards homogeneous coupled cell systems. In this class, we may define feedforward structure equivalently in a purely algebraic and in an order-theoretic formulation (Theorem 8.35). We investigate the effects of feedforward structure in its multiple definitions on the corresponding monoid representations in the case of one-dimensional internal dynamics. In particular, Theorem 9.2 provides a decomposition of the representation space into indecomposable components. Then we combine all insights to classify generic 1-parameter steady state bifurcations for feedforward networks via hands-on analysis. As before, we observe the amplification effect for our class of feedforward networks. However, due to the more complicated interaction structures, the picture becomes more complex. In particular, some expected amplifying branches may not exist. The main results are Theorems 10.8, 10.20 and 10.21 and Proposition 10.27, which classify all possible branching steady states in different scenarios and characterize the amplification effect. They are from the preprint [84]. Furthermore, the investigation of algebraic properties of the monoid representations related to feedforward networks have led to some interesting purely algebraic results in the preprint [82] that are not included in this thesis. Eventually, in Chapter 11, we apply the results on networks with high-dimensional internal dynamics from Chapter 7 to feedforward networks and obtain the classification of generic 1-parameter steady state bifurcations in such systems with high-dimensional internal dynamics as Theorem 11.12. In particular, we see that the amplification effect is independent of the dimension of the internal dynamics. This is part of the preprint [83].

Organization of the thesis

The conclusion of the introductory part (Part I) is made by a brief survey over different formalisms and approaches to network dynamical systems in Chapter 2. We provide some insight into early work (Section 2.1), introduce the groupoid formalism (Section 2.2) and the essentially equivalent combinatorial approach (Section 2.3). Then we review the introduction of graph fibrations into network dynamics (Section 2.4) and how this includes hidden symmetries into homogeneous coupled cell systems with asymmetric inputs (Section 2.5). As these objects are the main object of research in this thesis, the introduction will be very short and we will introduce them thoroughly at a later point in the text. Finally, we give an overview over the newer approaches in asynchronous networks (Section 2.6) and open systems (Section 2.7).

Part II investigates the implications and effects that representation theory of monoids and semigroups has on generic bifurcations in homogeneous coupled cell systems via hidden symmetries. To that end, we rigorously introduce the formalism and its major results in Chapter 3 as well as the necessary background on representation theory in Chapter 4. These also include several additional results that are not contained in the original publications. The first main result of this thesis is Theorem 5.11 on generic 1-parameter steady state bifurcations in monoid equivariant dynamics, which is proved in Chapter 5. We start in Section 5.1 by investigating nilpotent endomorphisms of representations that are direct sums of subrepresentations which are all pairwise isomorphic. These form building blocks of arbitrary representations and are called *isotypic components*. In Section 5.2 we complete the proof by reducing the question of generic generalized kernels in an arbitrary representation to that of determining the nilpotent endomorphisms of its isotypic components. Some technical details on submanifolds of matrix spaces are postponed to the appendix (Appendix A). We then summarize the generalizations to l-parameter bifurcations from Nijholt and Rink [77] in Section 5.3. In Chapter 6 we discuss the implications of Theorem 5.11 and its generalization for bifurcation analysis in homogeneous coupled cell systems with asymmetric inputs. Furthermore, we include an illustrating example in Section 6.2.

Then, we specifically focus on networks with high-dimensional internal dynamics in Chapter 7. We introduce a notation using tensor products to flexibly encode the attachment of a vector space to each cell without changing the network structure – which yields that the corresponding monoid representation is trivial on the internal phase spaces – in Section 7.1. This is used to investigate algebraic features for the main dimension reduction method (Theorem 7.12). In Section 7.2, we explore the implications for generic bifurcations in homogeneous coupled cell systems with high-dimensional internal dynamics. The results are summarized in Theorem 7.17 on 1-parameter steady state bifurcations, Theorem 7.18 on 1-parameter Hopf bifurcations, and in Theorems 7.22 to 7.24 on arbitrary l-parameter bifurcations.

In Part III, we combine the results from Part II in order to classify generic 1-parameter steady state bifurcations in feedforward networks. Chapter 8 explores a characterization of feedforward structure in a graphical interpretation and a purely algebraic formulation (Section 8.1) as well as an order theoretic formalism (Section 8.2). The equivalence of these approaches is proven in Theorem 8.35. Then, we exploit the algebraic definition in Chapter 9 in order to investigate properties of the monoid representation associated to a feedforward network. In particular, we find an algorithmic process to decompose the representation space into indecomposable subrepresentations (Theorem 9.6). In Chapter 10, we analyze generic steady state bifurcations in an arbitrary feedforward network with one-dimensional internal dynamics by hand. This is done by exploiting the specific structure that is forced upon the equations by the network. The sections in this chapter describe different technical scenarios that have to be taken care of separately. The classification of all possible branching steady state solutions is summarized in Theorems 10.8, 10.20 and 10.21. Some of the computational details are postponed to Appendix B. Furthermore, Proposition 10.27 characterizes the amplification effect in these bifurcations. Finally, in Chapter 11, we apply results and techniques for networks with high-dimensional internal dynamics for Chapter 7 to feedforward networks. This allows for a translation of the classification results in networks with one-dimensional internal dynamics to those with internal dynamics of arbitrary dimension (Theorem 11.12). We illustrate the result in an example of a 5-cell feedforward network with one-dimensional internal dynamics in Chapter 12.

We conclude the main body of work with some conclusive remarks in Part IV. Chapter 13 discusses that the inherently algebraic nature of networks as modular objects reflects severely in the methods brought forward in the investigation of network dynamics. We conclude in Chapter 14 with a short outlook on how this might be extended to incorporate more structural properties of networks in an algebraic framework.

Declaration of the author's contributions

The work on the PhD-project "Genericity in network dynamics" under the supervision of Prof. Dr. Reiner Lauterbach at the Department of Mathematics of the Universität Hamburg has led to the following notable publications and preprints the results of which are included in this thesis as laid out in the previous sections.

(a) [97] S. Schwenker. „Generic Steady State Bifurcations in Monoid Equivariant Dynamics with Applications in Homogeneous Coupled Cell Systems". *SIAM Journal on Mathematical Analysis* 50.3 (2018), pp. 2466–2485

(b) [84] E. Nijholt, B. Rink, and S. Schwenker. „Structural Properties and Bifurcations of Generalized Feedforward Networks". Preprint (2019)

(c) [83] E. Nijholt, B. Rink, and S. Schwenker. „Homogeneous Coupled Cell Systems with High-dimensional Internal Dynamics". Preprint (2019)

(d) [82] E. Nijholt, B. Rink, and S. Schwenker. „A new algorithm for computing idempotents of R-trivial monoids" (2019). arXiv: `1906.02844`
(not included directly, see Remark 9.9)

As mentioned before, the main result of (a) states that generic steady state bifurcations in 1-parameter families of monoid equivariant vector fields occur along an absolutely indecomposable subrepresentation. A generalization to l-parameter families as well as to Hopf bifurcations is provided by Nijholt and Rink [77]. Note that, even though this research was developed by the co-authors of (b) - (d), the work on (a) and [77] was entirely independent. As a matter of fact, both results were completed at almost exactly the same time in the spring of 2017 which does not reflect in the publication dates. This has led to increased communication eventually culminating in the decision to join forces on a project to tackle the most pressing questions on homogeneous coupled cell systems (and more general network structures). This still ongoing project was never focused on only one specific topic but ever evolving. It has led to the simultaneous development of the results of different scopes that are presented in (c) - (d) and in additional research that has not yet been completed. The intellectual property of these is shared equally between the three co-authors. The theories were developed in countless highly intense and long-lasting discussion sessions during mutual visits or at conference venues. Hence, no singular ideas are to be accredited precisely to individuals in hindsight. Also the process of producing the texts has been democratic and is not to be accounted for by only one of the co-authors.

Acknowledgments

First and foremost I would like to thank Prof. Dr. Reiner Lauterbach for his efforts as my supervisor and the enormous support he has provided me throughout the years. Ever since the first small projects leading to my Bachelor's thesis, he has been a great source of inspiration, introducing me to the areas of equivariant and network dynamics and other topics of recent development, and has always been able to provide valuable hints allowing me to proceed whenever I got stuck. But also countless discussions about topics far from the mathematical research world oftentimes allowed me to take a different or new perspective. Without him, I would not have worked in the field I am working in today and I would certainly not have been able to finish this thesis.

I am also highly grateful to my co-authors Prof. Dr. Bob Rink (Vrije Universiteit Amsterdam) and Dr. Eddie Nijholt (University of Illinois, Urbana-Champaign). After we established contact at the SIAM Conference on Applications of Dynamical Systems in the spring of 2017 they have quickly invited me to joint projects starting in the summer of 2017 and immediately shared their research ideas openly, which I have not taken for granted for a second. Our collaboration has been going on ever since and has proven to be very fruitful. The two of them have always been enormously supportive, also on a personal level, and I could not be more thankful.

Furthermore, I would like to express my gratitude towards my colleagues at the math department of the Universität Hamburg and in particular from the applied mathematics. Even if the topic of my research is slightly off the course of applied mathematics I have always experienced great interest and openness towards my work as well as the willingness to help wherever possible. This has laid the groundwork for the enjoyable working atmosphere in the last 4 years. In particular, I would like mention the members of the Dynamical Systems group during my time – Alex, Eugen, Florian, Haibo, Ivan, Jan Henrik, Mara, Maxim, Philipp – to which I had closest contacts for obvious reasons. Even more so, we have grown ever closer since I started my PhD in 2015, which makes the time particularly memorable.

I would also like to thank the Department of Mathematics, the MIN Graduate School International, and the Lorentz Center. They made it possible for me to attend the SIAM Conference on Applications of Dynamical Systems 2017 and the Workshop on Emerging Topics in Network Dynamical Systems 2017. Without these events, the cooperation with Prof. Dr. Bob Rink and Dr. Eddie Nijholt would be unthinkable and so would the contents of this thesis.

Finally, I am incredibly grateful to my friends and family. My parents have always been at my side and supported every decision that I made. Together with my family, my friends – the old ones from back in Bielefeld as well as the new ones from Hamburg – will always be there for me and provide stability and also a sometimes much-needed perspective from outside of academia. And last but not least, special thanks go to Anij. Your love and support and the knowledge that you will always have my back enable me to keep my head up and get through all the ups and downs along the way with a smile on my face.

Chapter 2

Network dynamical systems

As laid out in the introduction (Chapter 1), the study of dynamical systems with an underlying network structure comes with severe mathematical challenges. Reasons for this are manifold but can be summarized in two major points. Firstly, network dynamical systems behave vastly differently from systems without any additional structure. Examples are plenty. The most striking, however, is the phenomenon of *synchronization*: multiple (or even all) agents of the network behave in unison. This can be observed in many applications such as simultaneous firing of neurons or collective behavior in groups of animals such as bird flocks. Also generalizations can be thought of (and increasingly gain interest), such as phase-locked dynamics – e.g. rotating waves – or localized synchronization – called *chimeras* – in networks of oscillators. However, all these features are next to impossible for dynamical systems without any additional structure. They can only be found in very specific systems. For a general dynamical system, however, one can not expect the observables to behave uniformly. Secondly, the theory of dynamical systems with all its powerful tools that were developed since the late 19th century are not tailor-made for the study of network dynamical systems. Therefore, many methods quite simply lose track of the additional structure. For example, it is a challenging task to characterize coordinate transformations that keep a given network structure intact (see for example GOLUBITSKY and STEWART [56]). Such coordinate changes are the main ingredient in computing for example a normal form and thus the qualitative behavior of a system. It also plays a pivotal role in determining the so-called *generic* behavior of a system with given properties. As the established machinery fails to respect the network structure as inherent property of the system it accordingly also fails to distinguish qualitative behavior that is due to said structure – such as synchronization patterns. There are various other, mainly technical reasons, that result in the same issue.

Summarizing, one is mainly able to investigate small networks and determine their dynamical behavior by hand. Yet, insight into the underlying mechanisms by which a network influences dynamics is still rather limited. Therefore, analytical questions such as determining behavior and changes therein remain challenging, which also yields problems in applications, as requirements like prediction and control are not satisfactorily resolved. Since the beginning of this millennium, when interest in network dynamical systems sparked, several approaches have been put forth to address these issues. Interestingly, all of which are, in one way or another, algebraic in nature. This, however, might come as no surprise. Networks, in any suitable definition, are inherently algebraic objects themselves. They consist of small parts that are connected to form bigger parts which are then again connected to even larger parts that eventually make up the whole network. This concept is referred to as *modularity* which is fundamental in algebra.

Chapter 2 serves as a brief survey over a number of those approaches. It is not aimed at providing a detailed introduction into different theories but rather to present their underlying ideas. The collection is ordered chronologically only where possible, as some of the developments happened simultaneously and even shared some of the authors. We begin in Section 2.1 by briefly reviewing some major steps in the development of the theory of network dynamics starting with its origins until the proposition of more systematic approaches. In Section 2.2 we include the *groupoid formalism* by GOLUBITSKY, STEWART et al. It provides a formal way to write down network dynamical systems and as a result allows to characterize all patterns of synchrony for a given network. The essentially equivalent approach via *symbolic dynamics* by FIELD et al. is recapitulated in Section 2.3. Section 2.4 contains an approach by DEVILLE and LERMAN that connects the groupoid formalism to category theory. The keyword here is *graph fibrations* which are used to exploit the modularity of a network directly, to gain insight into the global dynamics of the corresponding system. To provide a quick outlook, we mention *semigroup networks* or *homogeneous coupled cell systems with asymmetric inputs* developed by NIJHOLT, RINK and SANDERS in Section 2.5. It introduces a connection to equivariant dynamics via hidden symmetries, which serve as an excellent explanation for many of the staggering phenomena that were mentioned above. Furthermore, graph fibrations allow for an interpretation of networks as *coupled modular control systems* that form a beautiful connection from the hidden symmetry approach to the groupoid formalism. As our own research aims at further development of this approach, we keep Section 2.5 very brief and provide full details in Chapter 3. Finally, we briefly review two very recent approaches in Sections 2.6 and 2.7. *Asynchronous networks* by BICK and FIELD, that focus on the *function* of a network, and *modular systems* by SPIVAK et al., which interpret networks as systems of systems, both incorporate the modular nature of a network and provide additional flexibility in the type of dynamics that can be encoded in the model. Due to their novelty, the full potential of both of these approaches can not be foreseen yet.

2.1 Early work

In its early days research on network dynamics was not so much focused on the actual network part of it but rather on the more eye-catching behavioral feature of *synchronization*. Spectacular occurrences were observed throughout applications from nature and engineering so that mathematical explanations were sought. Most investigations of that time do not care about the specific interaction structures but rather include regular (i.e. all-to-all or arbitrary) or random coupling (see the more recent WATTS and STROGATZ [114] for a brief discussion of these coupling types). WINFREE [115] provides what has been called "*a breakthrough in the study of synchronization*" in O'KEEFFE, HONG, and STROGATZ [86]. The author derives a mathematical model of circadian rhythms, the mechanism that serves as inner clock throughout biological beings. Therein cells are modeled by identical systems of ordinary differential equations that exhibit a stable limit cycle, coining the term *oscillator*. A periodic – with respect to the limit cycle – *coupling* mechanism is introduced, allowing the cells to communicate with each other if the coupling is strong enough. This leads to spontaneous emergence of synchrony. Later on, this model was further simplified and the equations were rigorously solved by KURAMOTO [68, 69]. Using an averaging method it is shown that the dynamics of each cell can be displayed by a system on the unit circle $\mathcal{S}^1$ and as a result the dynamics of the entire network can be represented on the torus $\mathcal{T}^N$, where N is the number of oscillators – the averaging method yields a normal form of the original system. This model has since been referred to as the *Kuramoto model*. It has seen countless applications in biology and physics and has sparked enormous interest in the study of dynamical properties of coupled oscillators. Since then, numerous generalizations of the model – especially concerning the interactions – have been proposed and applied. See ACEBRÓN et al. [2] for a summary of the model and its generalizations. PIKOVSKIJ, ROSENBLUM, and KURTHS [89] extensively surveys research on synchronization and prominently includes the Kuramoto model and its implications for the field.

The area of coupled oscillators remains highly active to this day with results describing staggering and complex dynamical properties being published every year. The equations in question governing dynamics of coupled oscillators typically take a form like

$$\dot{\varphi}_i = 1 + \varepsilon g_i(\varphi_1, \ldots, \varphi_N) \quad \text{for} \quad \varphi_i \in \mathcal{S}^1, i = 1, \ldots, N \tag{2.1}$$

(see for example ASHWIN and SWIFT [17]). Each $\varphi_i \in \mathcal{S}^1$ encodes the state of one oscillator. Then g_i characterizes the influence of the state of the other oscillators on the one with index i and $\varepsilon \ll 1$ accounts for the coupling strength – hence the term *weak coupling*. Specific interaction structures can then be modeled via the interaction functions g_i.[1] Weak coupling is required to guarantee the stability of the limit cycle in the original system and hence the success of the averaging procedure.

Additionally, further generalizations to describe non-periodic dynamics in individual cells have been considered as well. If one is willing to drop the assumption of existence of a stable limit cycle in the uncoupled system of each cell and with it the resulting simplifications one encounters systems like

$$\dot{x}_i = F(x_i) + \varepsilon \sum_{j=1}^{N} b_{i,j} G(x_j) \quad \text{for} \quad x_i \in \mathbb{R}^d, i = 1, \ldots, N \tag{2.2}$$

(see PECORA and CARROLL [88]). Here the individual internal dynamics is governed by an arbitrary ordinary differential equation on a finite-dimensional vector space via the function F. In order to investigate synchronization one employs variational methods – additional properties guarantee the dynamical invariance of the synchronous manifold $\{x_1 = \ldots = x_N\}$. Such models have for example been used to describe synchronized chaos, where two cells synchronize in their chaotic behavior (PECORA and CARROLL [87]). Note that the functions governing the dynamics in (2.2) are the same for each cell – the same internal dynamics F, the same coupling G, and the same coupling strength ε. This reflects the fact that all cells are assumed to be of the same type, the network is *homogeneous*. On the other hand, this is not the case in (2.1), the network is *non-homogeneous, heterogeneous* or *inhomogeneous*. The matrix $B = \{b_{i,j}\}_{i,j=1}^{N}$ – also referred to as the *adjacency matrix* – encodes the interaction structure of the network. The form (2.2), despite allowing for much more general dynamics than for example the Kuramoto model, imposes restrictions on the dynamics in consideration – besides the homogeneity –, especially in form of the additive nature of the couplings. The formalism does, however, include different interaction structures. More precisely different networks can be encoded in the same framework, which impose varying dynamical effects. Whereas in the early coupled oscillator models all-to-all coupling was standard, it is now possible to impose a specific network structure for example via the coupling functions g_i in (2.1) or the adjacency matrix B in (2.2). WANG [113] reviews the implications of different network topologies on synchronization.

It became clear, that this degree of freedom renders a thorough analysis of dynamical phenomena beyond synchronization impractical. Even for a fixed underlying network structure, results often seemed out of reach. Hence, the community explored means to reduce the complexity and identified *symmetry* as a suitable candidate. The permutations of the cells of a network that leave its interaction structure intact imply strong restrictions in the form of equivariance on possible functions driving the dynamics of the entire network. For example consider a network with identical cells that are arranged in a ring with undirected next-neighbor coupling (Figure 2.1). The network exhibits dihedral $\mathbf{D}_N$-symmetry as any rotations and reflections leave the network intact. Note that it does not have more symmetry (at least for N large enough), for example full permutation symmetry $\mathbf{S}_N$, as an arbitrary permutation of two cells leads to a change in the cells from which they receive their inputs. These symmetries are reflected in the differential equations governing the dynamics of the network in Figure 2.1. In their most general form they are

$$\begin{aligned}
\dot{x}_1 &= f(x_1, x_2, x_N) \\
\dot{x}_2 &= f(x_2, x_3, x_1) \\
&\vdots \\
\dot{x}_N &= f(x_N, x_1, x_{N-1}).
\end{aligned} \tag{2.3}$$

[1] In the original Kuramoto model the equation is similar. The internal dynamics is governed by a constant ω_i instead of 1 and the coupling is restricted to $\varepsilon g_i(\varphi_1, \ldots, \varphi_N) = \frac{K}{N} \sum_{j=1}^{N} \sin(\varphi_j - \varphi_i)$ for some constant K.

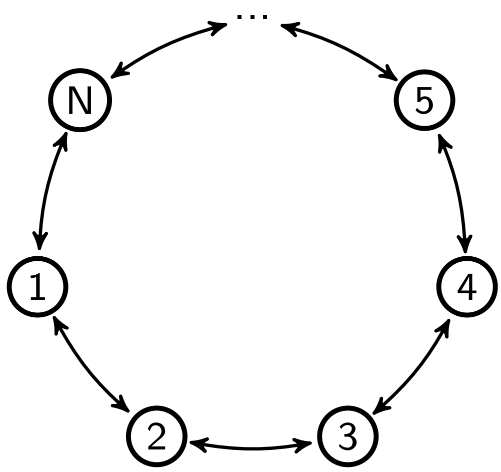


Figure 2.1: A homogeneous undirected N-cell ring

Applying any rotation or reflection on the indices of the cells leaves the set of differential equations unchanged. Using methods from equivariant bifurcation theory, it can be seen that, via Hopf bifurcation, such systems exhibit *phase locked* dynamics. That is, the cells oscillate with the same waveform but with regular phase shifts throughout the ring (see Golubitsky and Stewart [50]). This example is also considered in Golubitsky, Stewart, and Schaeffer [58], which is a milestone in the theory of equivariant dynamics (more background can be found for example in Chossat and Lauterbach [23] and Field [36]). The techniques were applied to a multitude of networks in applications (animal locomotion, Josephson junctions, binocular rivalry, chemical reactions, and more) and purely theoretic publications. Lots of interesting phenomena like *synchronization*, *phase-locking*, *partial synchrony*, *pattern formation*, *synchrony breaking bifurcations*, and others have been observed. See for example Alexander and Auchmuty [13], Aronson, Golubitsky, and Krupa [16], Ashwin and Swift [17], Diekman and Golubitsky [29], Dionne, Golubitsky, and Stewart [30, 31], Golubitsky and Stewart [52], and Hadley, Beasley, and Wiesenfeld [62]. Interestingly, however, the emergence of such dynamical features is not restricted to symmetric networks. When symmetry assumptions are violated similar dynamics can still be observed, most notably in the form of synchrony and synchrony breaking, highly complex and unexpected bifurcations but also phase locking. According to Golubitsky and Stewart [54] credit for this discovery belongs to Pivato, who around 2002 described a 16-cell network with four subsets of nodes, each exhibiting rotating wave solutions related to $\mathbb{Z}_4$-symmetry, even though the network has only trivial symmetry. A nice collection of further examples can be found in Golubitsky, Nicol, and Stewart [44]. This new development called for more precise approaches to networks and the implications of their structural properties on dynamics. The last 15 years have seen intense activity to that end, leading to the development of several formalisms. We use the upcoming sections to provide an overview over some of the resulting theories.

2.2 The groupoid formalism

Golubitsky, Stewart et al. made the crucial observation, that the dynamics of a cell is not directly influenced by all cells in the network but only by those, that have an arrow pointing at it. Hence, in order to understand the impact of the networks architecture or topology on its dynamics, one should investigate the dynamics 'locally' on what is called an *input set* of a cell to deduce global dynamical properties of the network. By comparing these input sets one gathers insight into which cells might behave similarly. This is done using algebraic objects called *groupoids*, which are used to to encode the local-global nature of the input sets (local: input sets; global: comparison thereof). The theory was developed in a series of papers (Golubitsky, Pivato, and Stewart [45], Golubitsky and Stewart

[53], GOLUBITSKY, STEWART, and TÖRÖK [59], and STEWART, GOLUBITSKY, and PIVATO [108]). Here we follow the presentation in [53].

The first step is to precisely define the object of study. We want to describe general networks of cells that exhibit general individual dynamics as well as general couplings. This is reflected in the fact that self-arrows and multiple arrows are allowed as well as that there are no restrictions on the functions governing the internal dynamics. However, means to classify nodes and arrows according to different *types* are included. In displays of networks as graphs, this is done by choosing various shapes for types of nodes and different colors for types of arrows. The formalism encodes these in the form of equivalence classes. Finally, note that all networks are assumed to be directed: cell p being influenced by cell q does not necessarily imply the converse. However, undirected networks are also covered by the formalism by including directed arrows both ways. We begin by restating the definitions of a network, the algebraic concepts relating to symmetry, and the vector fields that govern the dynamics of a network.

Definition 2.1 (Definition 5.1 in [53]). *A* coupled cell network $\mathcal{N}$ *consists of*

(i) a finite set $C = \{1, \dots, N\}$ *of* nodes *or* cells,

(ii) a finite set $E = \{e_1, \dots, e_n\}$ *of* edges *or* arrows,

(iii) an equivalence relation $\sim_C$ *on* C*, such that the equivalence class* $[p]_C$ *of a cell* $p \in C$ *describes its* type,

(iv) an equivalence relation $\sim_E$ *on* E*, such that the equivalence class* $[e]_E$ *of an edge* $e \in E$ *describes its* type *or* color,

(v) two maps $H, T \colon E \to C$*, such that* $H(e)$ *and* $T(e)$ *are respectively* head *and* tail *of* e *– i.e.* e *is an arrow from* $T(e)$ *to* $H(e)$ *–,*

and fulfills the consistency condition

$$e \sim_E \epsilon \quad \implies \quad H(e) \sim_C H(\epsilon), T(e) \sim_C T(\epsilon).$$

Definition 2.2 (Definitions 5.2 and 5.3 in [53]). *The* input set *of a cell* $p \in C$ *is the set of arrows pointing at it*

$$I(p) = \{e \in E \mid H(e) = p\}$$

and its elements are called input arrows *of* p*. Two cells are* input equivalent, $p \sim_I q$*, if and only if there exists a bijection* $\beta \colon I(p) \to I(q)$ *that preserves arrow types*

$$e \sim_E \beta(e) \quad \textit{for all} \quad e \in I(p).$$

Any such bijection is called an input isomorphism *from* p *to* q *and the set* $B(p,q)$ *denotes all input isomorphisms from* p *to* q*. The set of all input isomorphisms in the network*

$$\mathcal{B}_{\mathcal{N}} = \bigsqcup_{p,q \in C} B(p,q)$$

is the (symmetry) groupoid *of the network* $\mathcal{N}$*.*

Remark 2.3. The symmetry groupoid $\mathcal{B}_{\mathcal{N}}$ has the algebraic structure of a groupoid, as two input isomorphisms can be composed if and only if they are compatible with respect to input equivalence. That is, for $\beta \in B(p,q), \beta' \in B(p',q')$ we have $\beta \circ \beta' \in \mathcal{B}_{\mathcal{N}}$ if and only if $q' = p$. In particular, $\beta \circ \beta' \in B(p',q)$. △

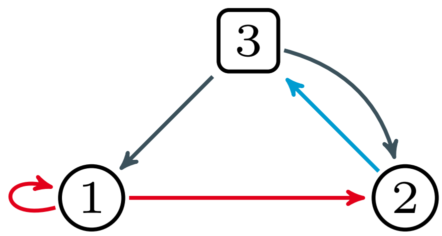


Figure 2.2: A 3-cell non-homogeneous network.

Note that input equivalence implies type equivalence due to the consistency condition. The groupoid encodes the way in which the inputs of one cell are 'the same' as those of another. The formalism can be employed to encode many examples of structural features of networks in a concise manner. For example a network is *homogeneous*, if and only if all its cells are input equivalent, i.e $B(p,q) \neq \emptyset$ for all $p,q \in C$. It is *regular*, if additionally all of its arrows are equivalent as well.

Example 2.1. Consider the 3-cell non-homogeneous network with two cell types and three arrow colors in Figure 2.2. Then cells 1 and 2 are input equivalent as both receive one **red** arrow and one **grey** arrow allowing us to bijectively map the self-loop of cell 1 onto the **red** arrow from cell 1 to 2. If for example the self arrow of cell 1 were of a different color, which does not violate the consistency condition, this would not be true. △

The next step is to introduce dynamics into these networks. As usual, these are governed by vector fields, which requires a notation of phase spaces and internal dynamics. These notions need to respect the structural features as they are encoded in the formalism. That is types of cells and arrows as well as the interaction structure as described via the equivalence relations and the heads and tails of edges. Every cell has its own state that is dynamically varied by an ordinary differential equation. Hence, we attach a finite-dimensional vector space V_p to a cell $p \in C$ which is called the *internal phase space*. Nodes of the same type exhibit the same type of dynamics – these are not necessarily equivalent – so that we require

$$p \sim_C q \quad \implies \quad V_p = V_q. \tag{2.4}$$

The *total phase space* is the phase space of the entire network and is defined as the direct product of all internal phase spaces

$$\prod_{p\in C} V_p.$$

In order to reflect the network's composition of individual cells, coordinates

$$x = (x_p)_{p\in C}$$

with $x_p \in V_p$ are chosen. More generally, one distinguishes the parts of the phase space that have an influence on a given cell p. In particular choose an ordering $I(p) = (e_1, \dots, e_s)$ and let $T(I(p)) = (T(e_1), \dots, T(e_s))$ be all cells that p receives an input from ordered accordingly. Note that the same cell might appear multiple times in $T(I(p))$, if p receives multiple inputs from the same cell. The *coupling phase space* and its coordinates are

$$V_{T(I(p))} = \prod_{j=1}^{s} V_{T(e_j)}, \quad x_{T(I(p))} = (x_{T(e_j)})_{j=1}^{s}.$$

In order to include the different types of cells and arrows into the dynamics, one describes the effect of input equivalence on coupling phase spaces in terms of pullback maps. That is, for $p \sim_I q$ and

$\beta \in B(p,q)$, we have

$$\beta^* \colon V_{T(I(q))} \to V_{T(I(p))}, \qquad (\beta^* x)_{T(e)} = x_{T(\beta(e))} \quad \text{for all} \quad e \in I(p).$$

Dynamics on networks are then governed by vector fields according to the following definition

Definition 2.4 (Definition 6.1 in [53]). *A vector field $F \colon \prod_{p \in C} V_p \to \prod_{p \in C} V_p$ is $\mathcal{N}$-*admissible*, if*

(i) the p-th component function $f_p(x)$ depends only on the internal phase space variables and the coupling phase space variables of cell p, meaning there is $\hat{f}_p \colon V_p \times V_{T(I(p))} \to V_p$, such that

$$f_p(x) = \hat{f}_p(x_p, x_{T(I(p))});$$

(ii) for all $\beta \in B(p,q)$

$$\hat{f}_q(x_q, x_{T(I(q))}) = \hat{f}_p(x_q, \beta^* x_{T(I(q))}) \quad \textit{for all} \quad x \in \prod_{p \in C} V_p.$$

When it is obvious from the context which network is considered, we also only speak of admissible vector fields. Note that these are entirely defined by one coordinate function for each input equivalence class due to the second condition. Furthermore it guarantees, that the state variables of inputs via the same arrow color are passed into the same entries of the internal function, hence, providing the same coupling influence. In that sense, the dynamical behavior of two cells of the same type that receive the same inputs are governed by the same mechanisms. Finally, we call a system of ordinary differential equations governed by an admissible vector field a *coupled cell system*. Even if the conditions in the definition appear rather algebraic and complicated, these are precisely the ones one would intuitively define for a given network structure.

Remark 2.5. The formalism allows for cells of the same type to receive different types of inputs. As a result their dynamics are governed by different functions. However, in many applications and in other formalism, oftentimes, one requires that cells of the same types receive the same types of inputs so that their dynamics are truly governed by the same mechanisms. In that case the cell type relation $\sim_C$ and the input equivalence relation $\sim_I$ are equal. On the other hand, by a slight abuse of notation, one could artificially relabel cell types according to input equivalence as only cells of the same type can be input equivalent. △

Example 2.2. We revisit the example in Figure 2.2. Once an ordering of the input arrows for each cell is chosen, the admissible vector fields are of the form

$$F(x) = \begin{pmatrix} f_1(x_1, x_1, x_3) \\ f_1(x_2, x_1, x_3) \\ f_3(x_3, x_2) \end{pmatrix}.$$

△

Example 2.3. If we slightly adapt the network from Figure 2.2 to include multiple arrows of the same color as in Figure 2.3, condition (ii) does not make any restrictions on the order of the inputs via the same arrow colors. The corresponding input function is invariant to permutations of these variables:

$$F(x) = \begin{pmatrix} f_1(x_1, x_1, x_2, x_3) \\ f_1(x_2, x_1, x_2, x_3) \\ f_3(x_3, x_2) \end{pmatrix},$$

with $f_1(x_1, x_1, x_2, x_3) = f_1(x_1, x_2, x_1, x_3)$. This is indicated by an line over the corresponding arguments

$$F(x) = \begin{pmatrix} f_1(x_1, \overline{x_1, x_2}, x_3) \\ f_1(x_2, \overline{x_1, x_2}, x_3) \\ f_3(x_3, x_2) \end{pmatrix}.$$

△

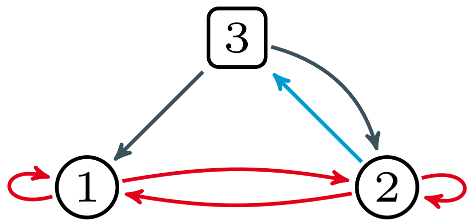


Figure 2.3: A 3-cell non-homogeneous network with multiple arrows of the same color.

Remark 2.6. Some examples or applications require a model that has internal dynamics that takes place on more complicated structures. For example recall that the internal phase space in coupled oscillators is often assumed to be the unit circle $\mathcal{S}^1$. These types of dynamics can be encoded in the groupoid formalism with some extra work as well. However, we chose not to include these parts here. △

With the formalism complete, it can be used to systematically investigate dynamical implications of network structure. As a matter of fact, it has been proven to be very powerful in that regard, most importantly in the analysis of patterns of synchrony that are inherent to a given network structure. That is, all coupled cell systems for that network support these patterns. This is also referred to as *robustness*. It can be shown that virtually all synchrony patterns can be encoded in equivalence relations on the cells that are called *balanced colorings*. In the remainder of this section, we briefly summarize the definition and the major results in that context.

A balanced coloring is a refinement of input equivalence. One assigns a color to each cell of the network. If the input sets of two cells of the same color – more precisely the cells from which they receive their inputs – are of the same color, while also respecting arrow colors, this coloring is called balanced.

Definition 2.7 (Definition 7.1 in [53]). *A* coloring *of the cells induces an equivalence relation $\bowtie$ on C via*

$$p \bowtie q \qquad \Longleftrightarrow \qquad p \text{ has the same color as } q.$$

This coloring is called balanced, *if and only if for all $p \bowtie q$ there exists an input isomorphism $\beta \in B(p, q)$ such that*

$$T(e) \bowtie T(\beta(e)) \quad \text{for all} \quad e \in I(p).$$

On the other hand, colorings can be used to indicate patterns of synchrony. To that end define the *polysynchronous subspace* of the total phase space corresponding to a coloring $\bowtie$ as

$$\Delta_{\bowtie} = \left\{ x \in \prod_{p \in C} V_p \;\middle|\; p \bowtie q \implies x_p = x_q \right\}.$$

Such a subspace describes all states in the total phase space in which two cells are synchronous when they are of the same color. In the same way we may reverse the order and define a coloring of cells according to a pattern of synchrony. A synchrony subspace is called *robust*, if it is flow-invariant under *all* admissible vector fields. The same term is used for the corresponding pattern of synchrony which is then inherent to the network structure. The relation between robust synchrony and balanced colorings is surprisingly powerful for the understanding of patterns of synchrony in dynamics of a network. We summarize some of the most staggering results informally in a similar spirit as in Golubitsky and Stewart [54]. Some of these statements were only proven under some technical conjectures in Golubitsky and Stewart [53]. However, since then a series of papers has successfully closed those gaps. Most notably we mention GOLUBITSKY, ROMANO, and WANG [47, 48] and STEWART and PARKER [109, 110] and GOLUBITSKY and STEWART [55] for a survey.

Note that, apart from patterns of synchrony, also the existence of hyperbolic equilibria and hyperbolic periodic solutions as well as their phase relations are dynamical features that can be inherent to the network structure. In particular, one of these phenomena is said to be *rigid*, if it exists for a given admissible vector field and persists under all small enough admissible perturbations. Let $y = (y_p)_{p \in C} \in \prod_{p \in C} V_p$ be a rigid steady state. Then it induces a coloring of cells according to

$$p \bowtie_y q \qquad \Longleftrightarrow \qquad p \sim_C q \text{ and } y_p = y_q.$$

This induces the corresponding *pattern of synchrony associated to* y denoted by $\Delta_{\bowtie_y}$. Note that similar definitions can be made for synchronous and phase related cells in a rigid periodic orbit. The associated pattern of synchrony is also said to be rigid.

Theorem 2.8 (Theorem 8.2 in [54]). *(i) A synchrony pattern is robust, if and only if the corresponding coloring is balanced.*

(ii) In a path-connected network[2] the pattern of synchrony associated to a hyperbolic equilibrium is rigid, if and only if the corresponding coloring is balanced.

(iii) In a path-connected network the pattern of synchrony associated to a hyperbolic periodic solution is rigid, if and only if the corresponding coloring is balanced.

Hence, patterns of synchrony that are inherent to a specific network structure, can be classified by balanced colorings.

Example 2.4. The network in Figure 2.2 allows for two balanced colorings determined by $1 \bowtie_1 2$ and $1 \not\bowtie_2 2$. Hence, the only non-trivial pattern of synchrony that can emerge is given by $x_1 \equiv x_2$. △

We close this section with a more in-depth investigation of synchronous dynamics. Once, the relation between synchrony and balanced colorings is established, one can ask the question, which dynamical features are possible *within* one pattern of synchrony. As we have seen before, the corresponding polysynchronous subspaces are dynamically invariant for all admissible vector fields, so that we can investigate dynamics restricted to them. A powerful tool to that end are *quotient networks* (strongly related to *quotient graphs* from graph theory). A quotient network is a new network that emerges from the original graph by identifying all cells of the same color – or equivalently all synchronous cells – while making sure the interaction structure remains by inserting arrows in a concise manner. This construction makes use of the fact that balanced colorings are equivalence relations on the cells with a specific relation to the network arrows.

Definition 2.9 (p. 337 in [53]). *Let $\bowtie$ be a balanced coloring on the network $\mathcal{N}$. Define the* quotient network $\mathcal{N}_{\bowtie}$ *as follows*

(i) The set of cells is

$$C_{\bowtie} = C/\bowtie = \{\overline{p} \mid p \in C\},$$

where $\overline{p}$ is the $\bowtie$-equivalence class of p.

(ii) Let $S \subset C$ be a set of representatives of the $\bowtie$-equivalence classes. The input arrows $\overline{e} \in I(\overline{p})$ of a cell $\overline{p}$ for $p \in S$ are identified with those of p: $\overline{e} = e$. This can be interpreted as a projection of arrows. The set of arrows of the quotient network is the disjoint union of all input sets

$$E_{\bowtie} = \bigsqcup_{p \in S} I(\overline{p}).$$

[2] A network in which there is a directed path in between any two cells.

(iii) *The arrow type relation is given by the arrow type relation of the original network*

$$\overline{p} \sim_{C_{\bowtie}} \overline{q} \quad \iff \quad p \sim_C q.$$

(iv) *The arrow color relation also follows from the arrow color relation of the original network*

$$\overline{e} \sim_{E_{\bowtie}} \overline{\epsilon} \quad \iff \quad e \sim_E \epsilon.$$

Note that due to the facts that the coloring is balanced and that a coloring is a refinement of the cell type relation, the definitions of the relations in the quotient network are well-defined and the consistency condition is fulfilled.

In order to understand the dynamical implications of the quotient network, it is worth noting that it emerges by identifying cells that are in synchrony. This mirrors the intuitive approach to the investigation of the dynamics on the corresponding subspace of reformulating the functions to only include each argument once. As a matter of fact, the quotient network construction pairs well with the definition of admissible vector fields.

Theorem 2.10 (Theorem 9.2 in [53])**.** *Let $\bowtie$ be a balanced coloring on the coupled cell network $\mathcal{N}$.*

(i) *The restriction of a $\mathcal{N}$-admissible vector field to $\Delta_{\bowtie}$ is $\mathcal{N}_{\bowtie}$-admissible.*

(ii) *Each $\mathcal{N}_{\bowtie}$-admissible vector field on the total phase space of the quotient network lifts to an $\mathcal{N}$-admissible vector field on the total phase space of the original network.*

Hence, in order to investigate dynamical features in the pattern of synchrony given by the balanced coloring $\bowtie$, it suffices to investigate admissible vector fields of the (in general) simpler quotient network. In particular, this has immediate consequences on the dimension of the problem. Additionally, note that multiple networks might yield the same quotient networks by a suitable pattern of synchrony. Hence, results for one network might be translated to another.

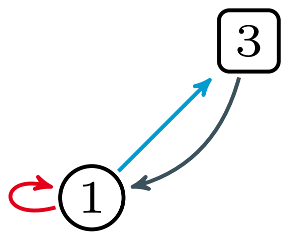


Figure 2.4: The quotient network with respect to $\bowtie$ of the network in Figure 2.2.

Example 2.5. The network in Figure 2.2 admits only one nontrivial coloring $\bowtie$ in which cells 1 and 2 are of the same color and cell 3 is of a different color. The corresponding quotient network is given in Figure 2.4. The admissible vector fields are

$$G(x_1, x_3) = \begin{pmatrix} g_1(x_1, x_1, x_3) \\ g_3(x_3, x_1) \end{pmatrix},$$

which can be lifted to admissible vector fields of the original network restricted to the synchrony subspace $\left\{x \in \prod_{p\in C} V_p \mid x_1 = x_2\right\}$:

$$F(x_1, x_1, x_3) = \begin{pmatrix} f_1(x_1, x_1, x_3) \\ f_1(x_1, x_1, x_3) \\ f_3(x_3, x_2) \end{pmatrix}.$$

$\triangle$

The one-to-one correspondence between robust synchrony balanced colorings has for example been exploited to derive a beautiful result relating phase relations in rigid periodic orbits to the network structure. It expresses possible phase relations in terms of a symmetry group of the quotient network with respect to the balanced coloring associated to the periodic orbit.

Theorem 2.11 (Theorem 8.3 in [54])**.** *In a path-connected network with a rigid periodic orbit of period Θ the phase difference in phase related cells is given by an integer multiple of Θ/k. To determine k, identify all rigidly synchronous cells. The corresponding quotient network has the cyclic symmetry group $\mathbb{Z}_k$.*

Note that, despite its power and use, in particular for the determination of synchrony, the groupoid formalism has not yet led to a general theory of synchrony breaking bifurcations. Assuming the existence of a fully synchronous equilibrium, one would like to characterize generic (i.e. inherent for a given network) transitions of such a solution when it changes its stability properties. Lots of interesting and highly unusual bifurcations have been described (see for example GOLUBITSKY and POSTLETHWAITE [46] for Hopf bifurcations with anomalous growth of the amplitude) and attempts have been made to set up general results. Examples include a linear theory for the deduction of spectral properties needed for the classification of branching solutions (GOLUBITSKY and LAUTERBACH [42]) or the development of lattice theoretic tools to determine existence and directions of branching solutions from a lattice of subspaces of partial synchrony (KAMEI [65, 66] and STEWART [106]). However, as is already indicated in Sections 11 and 16 in GOLUBITSKY and STEWART [53], a general theory is hard to develop. Some additional comments can be found in GOLUBITSKY and STEWART [55]. An issue lays in the generality of the groupoid formalism – almost any type of network can be realized in this framework. While the groupoid formalism incorporates a natural extension of group symmetries, there is no representation theory of groupoids that can be extended and, hence, no equivariant dynamics theory for groupoids either. As a result, in order to obtain general results on synchrony breaking bifurcations, one often restricts to smaller classes of networks, as we will see in the remainder of this thesis.

2.3 A combinatorial formulation of network dynamics

Around the same time as GOLUBITSKY, STEWART et al. worked on the groupoid inspired algebraic approach to network dynamics (Section 2.2), FIELD et al. developed their own formalism to investigate coupled dynamical systems (see AGARWAL and FIELD [3, 4], AGUIAR et al. [5], and FIELD [35]). The objective of their work – or the point of view of it – is slightly different. Instead of building a concise algebraic framework to capture the structural properties of networks they focused on the combinatorial nature. While the groupoid formalism encodes a given network structure and can be used for its dynamical analysis – the groupoid determines the algebraic properties needed for the investigation –, the combinatorial approach is geared towards the construction of networks with specified dynamical properties. Interestingly, the resulting formalisms are largely equivalent. In particular, they generate the same classes of dynamical systems. A discussion of this equivalence as well of the differences can be found already in the first publication FIELD [35]. Some additional details are stated in AGUIAR et al. [5]. In our presentation, we mainly follow the slightly more compact formulations in [5] which is closer to the groupoid formalism. In order to highlight the different scopes – as well as some special results – we extend it by some parts of the original publication FIELD [35]. However, due to the similarity to the groupoid formalism, we do not include as many details as in Section 2.2.

The basic objects of study in this approach are once again the cells of a network. Similar to conventions in electrical and computer engineering, these are considered to be *black boxes*, that receive certain inputs (of different types) and produce one output (determining the type of the cell). The output of one cell can then be connected to the input of another cell via so-called *patchcords*, when the type of input and output agrees (this can also be reflected in arrow colors). The network in Fi-

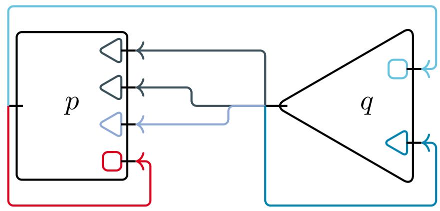


Figure 2.5: A symbolic network with 2 cells patched together.

gure 2.5 consists of two cells each of a different type or class. Each class fixes the types of inputs corresponding cells can receive. Given these input types there is a consistent way of patching the cells together by solving the corresponding combinatorial problem. However, if multiple cells of one type exist, one has more than one choice. This becomes important in the investigation of equivalence of networks. Hence, additional focus is laid upon cell classes. It is worth mentioning that these correspond to input equivalence classes in the groupoid formalism.

Definition 2.12 (Definition 2.5 in [5])**.** *Let $\mathfrak{C} = \{C_1, C_2, C_3, \dots\}$ be a set of* cell classes*. A* coupled cell network $\mathcal{N}$ *modeled on $\mathfrak{C}$ consists of a finite number of cells with classes in $\mathfrak{C}$ such that*

(i) the cells are patched *together according to the combinatorial restrictions imposed by the input-output type relation and*

(ii) there are no unfilled inputs.

In order for this definition to make sense, the set of classes needs to be *consistent*, i.e. the input types of each class must again be in $\mathfrak{C}$. Note that the interaction structure of the network can equivalently be characterized in terms of matrices. In AGUIAR et al. [5], these are called *adjacency matrices*.[3] A similar construction – the *connection matrix* – is used in FIELD [35]. After choosing a labeling of the cells $\{p_1, \dots, p_N\}$, one defines an $N \times N$ adjacency matrix for each input type (arrow color) as follows: the entry B^k_{ij} of B^k is the number of inputs of type k cell j receives from cell i. These matrices were already used before in order to classify coupled cell networks (see for example AGUIAR and DIAS [6]).

The introduction of dynamics into the framework is equivalent to that in the groupoid formalism. One attaches an internal phase space to each cell – most commonly a finite-dimensional vector field, but extensions are possible (see AGARWAL and FIELD [3, 4]) – and considers systems of ordinary differential equations. In order to respect the network structure, these depend on the state of the cell as well as on those that it receives an input from. Furthermore, cells of the same type are governed by the same functions, which are invariant under permutations of inputs of the same type. This provides the same classes of (admissible) vector fields as in the groupoid formalism.

Remark 2.13. The formalism also includes systems exhibiting discrete dynamics via iteration of admissible maps. Even more so, also *hybrid dynamical systems*, where some cells evolve continuously in time while others experience discrete changes of their state, are described. This involves including the discrete time frame into a continuous one. For more details see the examples in AGUIAR et al. [5]. △

Example 2.6. Dynamics of the network in Figure 2.5 are governed by the system of ordinary differential equations

$$\begin{aligned} \dot{x}_p &= f_p(x_p, \overline{x_q, x_q}, x_q, x_p) \\ \dot{x}_q &= f_q(x_q, x_p, x_q). \end{aligned}$$

In a slight abuse of notation we have identified cells and cell classes in this formulation, as there is only one cell of each type. △

[3]This term is also used in graph theory. Here it is extended to include multiple types of interactions, though.

In contrast to the groupoid formalism special emphasis is laid upon the fact that the functions governing the dynamics of individual cells

$$\dot{x}_{p_i} = f_k\left(x_{p_i}, x_{p_{j_1}}, \ldots, x_{p_{j_{l_k}}}\right),$$

where p_i is of class C_k, can be regarded as parameter dependent vector fields – the parameters being the variables of the cells inputs $x_{p_{j_1}}, \ldots, x_{p_{j_{l_k}}}$ – shifting the focus more towards the dynamics of an individual cell. In the original formulation (FIELD [35]) cells are even considered to be synonymous to their dynamics, which are defined before the notion of a network is made precise. Just as the explicit mechanism of constructing networks, this indicates a new interpretation of what the basic feature of a network is. Instead of considering a network to be an entity of individual nodes that are connected in a predefined way, the key feature is considered to be the constructive flexibility to take individual units and patch them together in several ways *to form new units*. This can also be seen in so-called *passive cells* in AGARWAL and FIELD [3, 4]. These cells with specific properties – providing specific transformations of their inputs – can be combined with existing cells in a network to form new ones. Similarly, if we loosen the consistency condition in the definition of a network by leaving inputs unfilled we can consider the entire network as a new cell with output computed from the outputs of its cells and inputs being the unfilled inputs of its cells. Informally speaking, we simply 'draw' a new cell shape containing the entire network. This structural property, which goes by the name *modularity*, is clearly closely related to the 'old' approach but provides a slightly different viewpoint that leads back to a question already formulated in the introduction (Chapter 1): Can the dynamical behavior of a network be understood from knowledge of its building blocks? This, however, is not thoroughly addressed in the literature using the combinatorial formalism of networks, which is legitimate as the formalism was not intended to answer these kinds of questions. Nevertheless, some results are available concerned with the construction of networks with specific dynamical properties (see AGUIAR et al. [5] and FIELD [37, 38] for networks exhibiting heteroclininc dynamics). On the other hand, some of the newer approaches (see Sections 2.4, 2.6 and 2.7) specifically construct networks in terms of modularity.

The combinatorial formalism is geared towards the question what influence a (given) network structure has on its dynamics. We close this section by briefly summarizing some of the main results in that regard. In particular, once again we are interested in dynamical features that are inherent to the network and independent of a specific dynamical system on it. For example, as the combinatorial and groupoid formalisms are equivalent, it may come as no surprise that robust patterns of synchrony – which are called *synchrony classes* here – are in one-to-one correspondence with *balanced* families of subsets of cells (Proposition 3 in FIELD [35]). These, in turn, are essentially balanced colorings from before.

On the other hand, a lot of work has been put into different means to classify networks. Several notions of *equivalence of networks* have been developed which allow to determine when two networks exhibit the same dynamics. This a non-trivial task, as can be seen for instance in Example 3.4 in AGUIAR et al. [5] where two networks with different architectures induce the same admissible vector fields – they are *ODE-equivalent*. However, in DIAS and STEWART [28] it is shown, that ODE-equivalence is equivalently determined on the linear level which greatly simplifies its investigation. Another notable notion is that of *patch-equivalence*, which is informally defined by the possibility to convert the network $\mathcal{N}$ into $\mathcal{N}'$ and vice versa by repatching the connections between cells according to the input-output type relation (Definition 3 in FIELD [35]). This is used to derive a *network normal form*. In contrast to the theory of dynamical systems, this is not a reformulation of the dynamical system in 'simpler' terms, but rather a method to construct a new network whose dynamics captures the essential features of the dynamics of the original network. Informally speaking, one repatches the connections in $\mathcal{N}$ to find a network $\widehat{\mathcal{N}}$ consisting of the 'smallest possible' transitive components that are pairwise not connected as well as their *slaved subnetworks* – i.e. subnetworks that do not provide inputs for any other cells. This network $\widehat{\mathcal{N}}$ is called the *normal form* of $\mathcal{N}$.

The normal form of $\mathcal{N}$ contains information about the 'minimal' – i.e. the least restrictive – classes of synchrony the network $\mathcal{N}$ supports. They are uniquely determined by the components of $\widehat{\mathcal{N}}$ and their slaved subnetworks. That is, in each connected component or in each slaved subnetwork all cells of the same type can be synchronized. As mentioned before, synchrony classes are in one-to one correspondence to balanced partitions. Hence, similar constructions can be made to investigate more general patterns of synchrony (see Section 6 in FIELD [35] and Section 3 in AGUIAR et al. [5]). In the groupoid formalism synchrony patterns were equivalently characterized by balanced colorings. Hence, the repatching approach provides a constructive viewpoint for this algebraic tool. For more details on network equivalence see AGARWAL and FIELD [3, 4], AGUIAR et al. [5], AGUIAR, DIAS, and RUAN [9], DIAS and STEWART [28], and FIELD [35].

2.4 Category theory and graph fibrations

Another approach to network dynamical systems that is inspired by both, algebraic relations as well as the modular structure of networks, was developed by DEVILLE and LERMAN [24, 25, 26]. Once again, the driving question is whether global dynamics of a network can be understood from knowledge of its building parts. We have seen this notion (explicitly) already in the combinatorial formalism (Section 2.3). On the other hand, this approach is also clearly inspired by the groupoid formalism and its approach by local symmetries – this is specifically mentioned in the introduction of DEVILLE and LERMAN [26]. However, one of the main objectives is to be able to encode more general types of dynamics in a consistent framework. Instead of setting up a formalism for ordinary differential equations on finite-dimensional real vector spaces that can be generalized to more general phase spaces and dynamics with extra work, they use techniques and notions from category theory to define networks of so-called *open systems* with dynamics on arbitrary manifolds. As a matter of fact, key results – as well as the general definition of network dynamical systems from before – follow from this approach as a special case. The category theoretical tools have the advantage of being able to describe dynamical systems coordinate-free which facilitates and simplifies proofs. For a comparison with the groupoid formalism see Remark 4.3.3 in DEVILLE and LERMAN [26]. A discussion of the notion of *modularity* and its realization in this approach as well as the combinatorial formalism can be found in the introduction of DEVILLE and LERMAN [25].

The most important feature of this theory is the notion of *graph fibrations*. These are maps between graphs or networks that are consistent with the interaction structure. In network dynamics graph fibrations provide semiconjugacies between admissible vector fields. Similar to the concept of equivalence of dynamical systems, such semiconjugacies allow to relate the dynamics of induced flows (this will be made more precise later).

In this summary we follow the most general formulation in DEVILLE and LERMAN [25]. We try to keep the presentation as simple as possible in order to transport only the main ideas. Therefore, we often use the language of category theory without precise definitions. New terms will be *highlighted* on first occasion and can be read as mere names for objects without detailed knowledge of their definitions and properties. For a more in-depth introduction, the reader should consult the mentioned references. The only thing to keep in mind is that a *category* is a collection of *objects* with *maps* in between. Furthermore, it helps to think of networks in the groupoid formalism in the version presented in Section 2.2 as examples for the objects defined here.

Network structure in general is, once again, encoded using (directed) graphs. As these are defined similarly to those in Definition 2.1, we slightly alter the notation to match the one used previously. Together with suitable maps in between two graphs, they form a category.

Definition 2.14 (Definitions 2.3, 2.4 and Remark 2.5 in [25]). *A* graph *or* network $\mathcal{N} = \{E \rightrightarrows C\}$ *consists of a finite set* C *of* nodes *and a finite set* E *of* arrows *as well as two maps* $H, T\colon E \to C$ *describing the* head *and* tail *(*source *and* target*) of an arrow. A* map of graphs/networks $\varphi\colon \mathcal{N}_1 \to \mathcal{N}_2$

consists of two maps $\varphi_C\colon C_1 \to C_2$ *and* $\varphi_E\colon E_1 \to E_2$ *such that*

$$\varphi_C(H(e)) = H(\varphi_E(e)), \quad \varphi_C(T(e)) = T(\varphi_E(e))$$

for all $e \in E_1$*. The collection of all graphs together with all maps of graphs forms a category* **Graph***.*

Remark 2.15. Contrary to the groupoid formalism, different types or colors (of nodes and arrows) are not distinguished in this framework. That does not mean, all cells and arrows are strictly of the same or of different types. This notion is simply not included in the framework. However, a colored version was proposed as well in DEVILLE and LERMAN [24]. △

With the definition of networks established, one introduces dynamics into the framework. Once again, this begins with a choice of possible internal phase spaces (i.e. possible states of each cell) and suitable interacting dynamical systems. Here the authors choose to allow for the most general types of dynamics governed by vector fields on manifolds – which might well be (finite or infinite-dimensional) vector spaces. This comes with severe challenges as the functions governing internal dynamics in this general setting do not form closed vector fields but rather so-called *open* or *control systems* known from control theory literature (see for example TABUADA and PAPPAS [111]). The category theory notation allows to avoid choosing a labeling of nodes as well as coordinates for the internal manifolds, which provides enough freedom to circumvent these obstacles. As we work with vector fields on manifolds, we employ the usual notation of denoting the *tangent bundle* of a manifold M by TM and the tangent space at a point $x \in M$ by T_xM.

Definition 2.16 (Definitions 2.6 and 2.12 in [25])**.** *A* network of manifolds $(\mathcal{N}, \mathcal{P})$ *is a pair of a network* $\mathcal{N}$ *and a function* $\mathcal{P}\colon C \to$ **Man**[4] *that assigns a manifold to each cell. A* map of networks of manifolds $\varphi\colon (\mathcal{N}_1, \mathcal{P}_1) \to (\mathcal{N}_2, \mathcal{P}_2)$ *is a map of networks* $\varphi\colon \mathcal{N}_1 \to \mathcal{N}_2$ *such that*

$$\mathcal{P}_2 \circ \varphi = \mathcal{P}_1.$$

Networks of manifolds form a category **Graph/Man***. The* total phase space *of a network of manifolds is defined as the* categorical product

$$\mathbb{P}\mathcal{N} \equiv \mathbb{P}(\mathcal{N}, \mathcal{P}) = \prod_{p \in C} \mathcal{P}(p).$$

The notation of a categorical product can be regarded as a generalization of the cartesian product to categories. These are well-defined in the categories in question here. Open dynamical systems are governed by vector fields that depend on additional control variables from another manifold. The idea is to assign an open system to each cell of a network and let the state of one be the control variable of another according to the interaction structure.

Definition 2.17 (Definition 2.23 in [25])**.** *For a manifold* M *of* state variables *and a manifold* U *of* control variables *define the vector space of* control systems *(or* open systems*)*

$$\textbf{Control}(M \times U \to TM) = \{f\colon M \times U \to TM \mid f(x,u) \in T_xM \textit{ for all } (x,u) \in M \times U\}.$$

The introduction of the interaction structure into collections of open systems proceeds in a similar fashion as in the groupoid formalism. However, it requires a lot more notation and machinery from category theory the details of which we omit here. An important aspect is a generalization of input sets (compare to Definition 2.2). The input tree encodes all inputs into a cell p in a new graph centered around p.

[4] **Man** is the category of smooth finite-dimensional manifolds and smooth maps.

Definition 2.18 (Definition 2.26 and Proof of Proposition 2.29 in [25]). *The* input tree $\mathcal{I}(p)$ *of a cell* $p \in C$ *is a graph with cells* C_p *and edges* E_p *defined as*

$$C_p = \{p\} \sqcup H^{-1}(p), \quad E_p = \{(p, e) \mid e \in E, H(e) = p\}.$$

Head and tail of an arrow (p, e) *are defined to be*

$$H(p, e) = p, \quad T(p, e) = e.$$

The total phase space of the input tree is defined as

$$\mathbb{P}\mathcal{I}(p) = \mathcal{P}(p) \times \prod_{e \in H^{-1}(p)} \mathcal{P}(T(e)),$$

i.e. we assign the internal phase space of the tail cell of each arrow pointing at p *to the corresponding edge.*

Locally, interactions can be considered as a control system on the input tree, i.e. an object of **Control**$(\mathbb{P}\mathcal{I}(p) \to \mathbb{P}p)$, where $\mathbb{P}p = \mathcal{P}(p)$ is the internal phase space of cell p – here, we slightly abuse the notation of DeVille and Lerman [25]. Then, one passes the interaction structure of a network of manifolds to a collection of open systems on the input trees

$$F = (f_p)_{p \in C} \in \mathfrak{Ctrl}(\mathcal{N}, \mathcal{P}) = \prod_{p \in C} \textbf{Control}(\mathbb{P}\mathcal{I}(p) \to \mathbb{P}p).$$

Interactions of the collection of open systems are realized by an interconnection map $\mathfrak{I}(F)$. In particular, $\mathfrak{I}(F)$ realizes a closed system on the total phase space. The precise construction of the map $\mathfrak{I}$ can be found in Theorem 2.32 in DeVille and Lerman [25]. Its main property described informally is a commutativity relation that guarantees the equality of each component of the closed system with the control system of the corresponding cell. For our purpose it suffices to consider it to contain the information which cells state variables serve as control variables for others. When restricting to networks subject to the groupoid formalism, this condition matches the groupoid invariance.

With this highly general framework for network dynamical systems in place one tries to exploit the structural properties of networks to deduce dynamical properties. In particular, the input structure of each cell imposes restrictions on possible dynamics. In the groupoid formalism, this observation was made precise for example in the characterization of robust synchrony in terms of balanced colorings. However, the authors realized, that more truth can be found / encoded in maps between networks. Maps that respect the inputs into each cell, called *graph fibrations*, are of particular interest. This can be seen as a generalization of the procedure of finding balanced colorings or balanced partitions in the previous approaches. Graph fibrations were not developed by DeVille and Lerman but were introduced in Boldi and Vigna [22]. However, their implications on network dynamics have not been investigated previously.

Definition 2.19 (Definition 3.1 in [25]). *A map* $\varphi\colon \mathcal{N}_1 \to \mathcal{N}_2$ *between networks is called a* graph fibration, *if for each node* $p \in C_1$ *and each arrow* $\epsilon \in E_2$ *pointing at* $\varphi_C(p)$, *that is* $H(\epsilon) = \varphi_C(p)$, *there is a unique arrow* $e \in E_1$ *pointing at* p, *that is* $H(e) = p$, *such that* $\varphi_E(e) = \epsilon$.

Graph fibrations have severe implications on dynamical systems. Recall that dynamical equivalence is defined by the existence of a map that intertwines two flows. That is, for two vector fields $F\colon M \to TM$ and $G\colon N \to TN$ with corresponding flows Φ_t and Ψ_t respectively does there exist an invertible map $h\colon M \to N$ such that

$$h \circ \Phi_t = \Psi_t \circ h \quad \iff \quad Dh \circ F = G \circ h,$$

where Dh is the differential of h.[5] If such a map or *conjugacy* exists, the dynamics induced by the two vector fields are in some sense the same – they are qualitatively the same. But also if we relax the invertibility condition of the conjugacy, dynamical features of one can be found within the other system. For example, if h is surjective, it sends integral curves of one system to integral curves of the other. We speak of a *semiconjugacy*. Interestingly, graph fibrations induce (in general non-invertible) conjugacies between the closed dynamical systems on total phase spaces. The precise formulation and proof of this result requires several intermediate steps, defining suitable categories and maps in between. Hence, we summarize only parts of it.

Theorem 2.20 (Theorems 3.8 and 3.11 in [25]). *A graph fibration $\varphi\colon \mathcal{N}_1 \to \mathcal{N}_2$ induces a linear map of control systems on networks of manifolds in the opposite direction*

$$\varphi^*\colon \mathfrak{Ctrl}(\mathcal{N}_2, \mathcal{P}_2) \to \mathfrak{Ctrl}(\mathcal{N}_1, \mathcal{P}_1).$$

Furthermore, it induces a map of total phase spaces $\mathbb{P}\varphi\colon \mathbb{P}\mathcal{N}_2 \to \mathbb{P}\mathcal{N}_1$ given by $\left(\mathbb{P}\varphi(x_q)_{q\in C}\right)_p = x_{\varphi(p)}$, which yields a conjugacy between corresponding closed dynamical systems $\mathfrak{J}_2(F_2)$ and $\mathfrak{J}_1(\varphi^* F_2)$, i.e.

$$D\mathbb{P}\varphi \circ \mathfrak{J}_2(F_2) = \mathfrak{J}_1(\varphi^* F_2) \circ \mathbb{P}\varphi. \tag{2.5}$$

The conjugacy between dynamical systems can be exploited to analyze dynamical features of network dynamical systems by relating different networks via graph fibrations. For example, for a surjective graph fibration $\varphi\colon \mathcal{N}_1 \to \mathcal{N}_2$ the induced map between total phase spaces is an embedding with the image

$$\Delta_\varphi = \mathrm{im}\,(D\mathbb{P}\varphi) = \{x \in \mathbb{P}\mathcal{N}_1 \mid x_p = x_q, \text{ if } \varphi(p) = \varphi(q)\},$$

(see Lemma 2.20 in DeVille and Lerman [25]). In the language of the groupoid formalism, this is a polydiagonal that describes a pattern of synchrony. Due to the conjugacy relation (2.5), each network dynamical system $\mathfrak{J}_1(\varphi^* F_2)$ respects this polydiagonal submanifold. Note that the map sending a network to its quotient by a balanced coloring is an instance of a surjective graph fibration. On the other hand, an injective graph fibration gives rise to a surjective submersion of total phase spaces (see Lemma 2.21 in DeVille and Lerman [25]). This can for example be used to encode feedforward-type structures in networks.

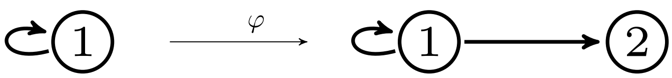


Figure 2.6: An injective graph fibration.

Example 2.7 (Compare to Example 5.2.2 in DeVille and Lerman [26]). In Figure 2.6 we depict an injective graph fibration φ of a network $\mathcal{N}_1$ consisting of a cell with a self-loop to a 2-cell feedforward chain $\mathcal{N}_2$. It maps cell 1 to 1 and the self-loop onto the self-loop. The corresponding conjugacy maps

$$\mathbb{P}\varphi(x_1, x_2) = (x_1)$$

which yields

$$D\mathbb{P}\varphi(x_1, x_2) = (x_1).$$

For any $F = (F_1, F_2) \in \mathfrak{Ctrl}(\mathcal{N}_2, \mathcal{P}_2)$ we have

$$\begin{aligned} \mathfrak{J}_2(F)(x_1, x_2) &= (F_1(x_1), F_2(x_1, x_2)) \\ \varphi^* F &= (F_1) \\ \mathfrak{J}_1(\varphi^* F)(x_1) &= (F_1(x_1)). \end{aligned} \tag{2.6}$$

[5] In the language of category theory h is a morphism in the category of dynamical systems.

We immediately see the conjugacy condition (2.5) is satisfied as

$$(D\mathbb{P}\varphi \circ \mathfrak{J}_2(F))\,(x_1,x_2) = F_1(x_1) = (\mathfrak{J}_1(\varphi^* F) \circ \mathbb{P}\varphi(x_1,x_2))\,.$$

On the other hand, any closed system on $\mathcal{P}_2\mathcal{N}_2$ needs to satisfy this conjugacy condition in order to respect the network structure of $\mathcal{N}_2$. In particular, a closed system can only be admissible if its component corresponding to cell 1 does not depend on that corresponding to cell 2. This matches the expectation given the graph has no feedback into cell 1. The semiconjugacy encodes the structural property that the dynamics of $\mathcal{N}_2$ is driven by cell 1. △

The example also provides a nice impression of the concept of modularity in this framework. An integral part of the structure of the larger network is the fact that it 'contains' the smaller one, which can be investigated individually. The usage of open systems – similar to the parameter dependent vector fields in the combinatorial approach (Section 2.3) – provides the tool to actually incorporate this structural property into a dynamical system. Further examples of network structure encoded in graph fibrations can be found in the mentioned references. Finally, we would like to mention, that also self-fibrations are possible, i.e. a graph fibration from a network to itself. Due to the conjugacy relation (2.5), these give rise to what can be regarded as a generalization of symmetry. This will be the main feature of Section 2.5. In general, it can be said, that the implications of graph fibrations for network dynamical systems are not yet fully understood. It remains to be seen, what kinds of structural properties of networks can be encoded as graph fibrations so that their impact can be investigated. Nevertheless, we are given a powerful tool to do just that in future research.

Remark 2.21. As we make use of this approach in the remainder of this thesis, we want to briefly remark some specifics about the colored version of this theory. In a straightforward manner, one introduces colors of cells and of arrows into the network (Definition 2.14) using compatible equivalence relations as in the groupoid formalism (see Definition 2.1). More importantly for us, a graph fibration of colored networks is a graph fibration as in Definition 2.19, that additionally respects colors. In particular, it sends cells and arrows to cells and arrows of the same color. This requires both networks to be colored with the 'same' colors – or at least the colors are identifiable. The crucial semiconjugacy result (Theorem 2.20) then holds for colored graphs as well. For more details on this see DeVille and Lerman [24]. △

2.5 Semigroup networks

A standard method to investigate dynamics without any additional structure locally – for example in the vicinity of an equilibrium – that has proven to be extremely useful in the analysis of local bifurcations and generic properties is that of a *normal form* (see for example Murdock [75]). That is, a 'simple' representation of the vector fields in consideration that includes all the interesting dynamical properties, which is derived via coordinate transformations. Inspired by this powerful tool, and in particular the way it can be computed in terms of Hamiltonian functions for Hamiltonian systems, Rink and Sanders, later joined by Nijholt, developed an approach to network dynamics of their own (see Nijholt, Rink, and Sanders [78] and Rink and Sanders [90, 91]). In order to compute a normal form in standard ways, one has to compute concatenations – more specifically Lie brackets – of admissible vector fields. As we already hinted at in the introduction (Chapter 1), this quickly leads to the problem that in general the interaction structure of a network is lost by doing so. The authors decided to restrict to a subclass of networks – *(homogeneous) coupled cell systems with asymmetric inputs* – and introduce the algebraic structure of a *semigroup* which allowed them to avoid this major issue. This formalism led to multiple interesting observations that could only be explained later in relation to techniques from the other formalisms that we introduced in the previous sections. As all the major results in this thesis are devoted to the development of this formalism, we only illustrate some of the main ideas and the chronology of progress in the form of a running example in this section. We provide a thorough introduction into semigroup networks in Chapter 3. Let us mention that the chronological presentation may seem to be wrong at first sight. This is due to the fact that the theory developed quickly so that results were published seemingly ‘out of order’.

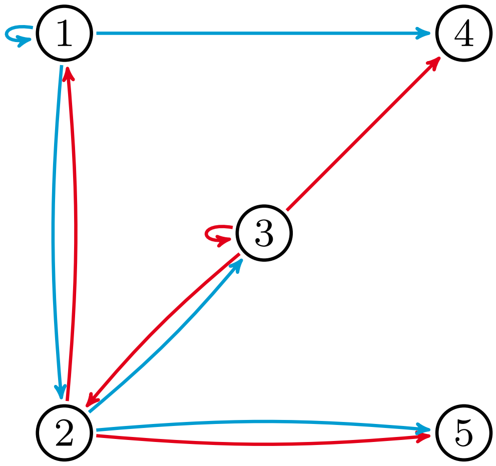


Figure 2.7: A 5-cell homogeneous coupled cell system with asymmetric inputs.

For the running example consider the homogeneous network $\mathcal{N}'$ in Figure 2.7 whose admissible vector fields are of the form

$$F(x) = \begin{pmatrix} f(x_1, x_2, x_1) \\ f(x_2, x_3, x_1) \\ f(x_3, x_3, x_2) \\ f(x_4, x_3, x_1) \\ f(x_5, x_2, x_2) \end{pmatrix}, \tag{2.7}$$

where we have attached the same finite-dimensional vector space to each cell. As was mentioned before, one would like to be able to derive dynamical features that are inherent to the network structure, in particular generic bifurcations, locally from a *normal form* of admissible vector fields. The standard procedure of its computation requires to derive terms from expressions including concatenations of vector fields. This leads to one of the fundamental problems in the investigation of network dynamics that we introduced already in the introduction (see (1.1)), which is that concatenations in general do not respect the interaction structure of a network. If we concatenate two admissible vector fields F and G of $\mathcal{N}'$ (as in (2.7)), we obtain

$$(F \circ G)(x) = \begin{pmatrix} f(g(x_1, x_2, x_1), g(x_2, x_3, x_1), g(x_1, x_2, x_1)) \\ f(g(x_2, x_3, x_1), g(x_3, x_3, x_2), g(x_1, x_2, x_1)) \\ f(g(x_3, x_3, x_2), g(x_3, x_3, x_2), g(x_2, x_3, x_1)) \\ f(g(x_4, x_3, x_1), g(x_3, x_3, x_2), g(x_1, x_2, x_1)) \\ f(g(x_5, x_2, x_2), g(x_2, x_3, x_1), g(x_2, x_3, x_1)) \end{pmatrix}.$$

This is no longer admissible, as, for example, the fourth entry does not only depend on the variables x_1, x_3 and x_4 as is dictated by the network structure, but also on x_2. From the technical perspective this has the effect that the admissible vector fields do not form a Lie algebra, which is needed for the computation of an admissible normal form.

In terms of combinatorics, we see that by concatenation the set of cells having an input into 4 changes, more precisely it enlarges. However, it does not do so arbitrarily. As a matter of fact, in the concatenation cell 4 depends on 2 but not on 5. The reason for this is that in the network 4 depends on 3 which in turn depends on 2. So 2 has what we call an *indirect input* into cell 4. On the other hand, cell 5 has no indirect input into 4. One of the key ideas the authors proposed was to regard all indirect inputs as direct inputs – basically by adding arrows for every indirect input – so as to make the class of admissible vector fields closed under concatenations. Note that this is not a trivial task. In general there is no concise way of adding arrows to a network while respecting cell and arrow types. For example, cell 4 has an indirect input but cell 2 does not. As both are of the same type, adding an arrow from cell 2 to cell 4 requires another arrow pointing at cell 2 as well. The question is, where does this input come from?

This issue can be avoided in networks with asymmetric inputs – every cell receives only one arrow of each color. Here we also restrict to homogeneous networks which makes the notation less heavy but is not crucial for the construction. The indirect input from cell 2 to cell 4 comes via a **red** arrow from cell 3 which receives a **blue** arrow from cell 2. Then we can determine similar 'indirect' inputs for the remaining cells in the same way. That is, starting at any cell, we first follow a **red** arrow backwards and then a **blue** one. This determines which cell passes information through that specific channel – i.e. an indirect input. All inputs derived from a **red** and a **blue** arrow in this way are then depicted by an additional arrow of a new color. Due to the asymmetry in the inputs, this procedure is well-defined and unique. More precisely it produces new *arrow colors* – consistent with the network structure – from existing ones, instead of only providing new arrows. We refer to the backwards following of arrows while respecting colors as the *(backward) concatenation of arrows*. This procedure can be continued to produce all possible new arrow colors – this involves also concatenating arrows of new colors. Note that the procedure stops at some point – when the network is finite –, if we identify arrow colors which describe the exact same interaction structure on all cells. That is, when all indirect inputs have been added as direct inputs as new arrow colors. The extended network $\mathcal{N}$ for $\mathcal{N}'$ is shown in Figure 2.8. Its admissible vector fields are of the form

$$\gamma_f(x) = \begin{pmatrix} f(x_1, x_2, x_1, x_1, x_2, x_3, x_1, x_2) \\ f(x_2, x_3, x_1, x_2, x_2, x_3, x_1, x_2) \\ f(x_3, x_3, x_2, x_2, x_3, x_3, x_1, x_2) \\ f(x_4, x_3, x_1, x_2, x_2, x_3, x_1, x_2) \\ f(x_5, x_2, x_2, x_1, x_3, x_3, x_1, x_2) \end{pmatrix}. \tag{2.8}$$

The interaction structure of the extended network is a lot more complex. As a matter of fact, already small networks with only a few arrow colors, may yield to a large collection of arrows in the extended network. However, it brings the advantage that the admissible networks are closed under concatenations resulting in the benefits hinted at before.

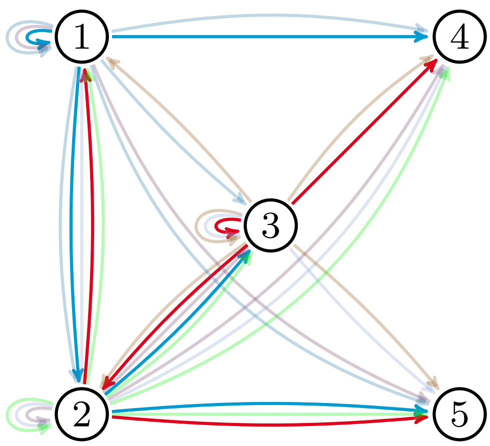


Figure 2.8: The (extended) semigroup network of $\mathcal{N}'$ in Figure 2.7.

To make the introduction slightly more complete, the mechanism of extending the network to include indirect inputs can be made precise in mathematical terms. To that end, we state the following definition.

Definition 2.22 (Sections 2 and 3 in [92])**.** *A* homogeneous network with asymmetric inputs $\mathcal{N}'$ *consists of a collection of* cells $C = \{1, \dots, N\}$ *and a collection of* input maps $\Sigma' = \{\sigma_1, \dots, \sigma_n\}$*. Each input map* $\sigma \colon C \to C$ *is a map on the set of cells that encodes one* arrow color *in the sense that* $p \in C$ *receives an arrow of color* σ *from the cell* $\sigma(p)$*. We attach a finite-dimensional real vector space* V *to each cell and consider* internal dynamics *governed by the same function* $f \colon V^{n+1} \to V$*. A* coupled cell system *or* network dynamical system *is a system of ordinary differential equations governed by* admissible vector fields *of the form*

$$\gamma_f(x) = \begin{pmatrix} f(x_1, x_{\sigma_1(1)}, \dots, x_{\sigma_n(1)}) \\ \vdots \\ f(x_N, x_{\sigma_1(N)}, \dots, x_{\sigma_n(N)}) \end{pmatrix}.$$

Note that the characterization of the interaction structure in terms of input maps is only possible because of the assumption of asymmetric inputs. For example the **red** arrows in Figure 2.7 are encoded in the map

$$\sigma \colon \{1, \dots, 5\} \to \{1, \dots, 5\}, \quad 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 3, 4 \mapsto 3, 5 \mapsto 2.$$

In general, as $\sigma \colon C \to C$, the input maps can be concatenated, which is the formalization of concatenation of arrows, i.e. $\sigma\tau = \sigma \circ \tau$ produces the new arrow color describing indirect inputs following a τ-arrow backwards and then a σ-arrow. The closure of Σ' under these concatenations – the smallest set closed under concatenations that contains Σ' – can be computed by adding all possible concatenated input functions. The resulting set has the algebraic structure of a *semigroup* – informally speaking a *group* without inverses and identity –, which we denote by Σ. It is the semigroup *generated* by Σ'. This semigroup is the set of input maps of the extended network $\mathcal{N}$. A network whose input sets form a semigroup is therefore called a *semigroup network*. The notation can be made even more compact. Note that we usually assume that all cells depend on themselves as well. We could have indicated this by attaching a (**black**) self-loop to each cell. The corresponding input map is the identity $\mathrm{Id}_C \colon C \to C$. Hence, we assume $\sigma_1 = \mathrm{Id}_C$ to be part of the generating set and adapt the admissible vector fields accordingly. In that case, Id_C also serves as an identity element in the semigroup which we denote by Id. Thus, Σ has the structure of a *monoid*. Note furthermore, that the class of networks defined here could have been encoded in terms of the groupoid formalism as well.

It turns out that the additional algebraic structure has enormous benefits for the computations of normal forms. As a matter of fact, concatenation of admissible vector fields can be expressed in terms of the internal functions and essentially boils down to multiplication of input maps. This yields a Lie algebra structure on the space of admissible vector fields that allows for the computation of normal forms according to the standard methods (see Rink and Sanders [92]). Together with the combinatorial interpretation of the network extension, this serves as an argument for the investigation of (extended) semigroup networks instead of homogeneous networks with asymmetric inputs. That is, to accept the semigroup extension of a network as its 'natural state'. The semigroup is an invariant of a network that carries its interaction structure and encodes it for investigations of its dynamic evolution. Additionally, note that the class of admissible vector fields of the original network is contained in that of the extended network. An admissible vector field for the original network can be extended by trivial dependence on the additional variables provided by indirect inputs. Finally, let us remark that the extension has no effect on robust synchrony whatsoever. That is, the original network and its extension have the same robust patterns of synchrony, the corresponding polydiagonals are invariant under the flows induced by all admissible vector fields for both networks (see Lemma 7.3 in Rink and Sanders [92]). In particular, normal forms computed for semigroup networks exhibit the same synchrony patterns as the original (non-extended) network. Hence, this crucial inherent property of a network is unaffected by the semigroup extension, which supports the idea that extended networks are a natural class of networks even further. The normal form of semigroup networks has successfully been applied to investigate unusual (amplified) Hopf bifurcations in feed-forward chains Rink and Sanders [90].

Another staggering property of semigroup networks was discovered by the authors as a byproduct of their investigations of normal forms in Rink and Sanders [92]. They realized that a purely algebraic construction produces a second network with interesting properties. It has as many cells as the semigroup network has input maps and the same input maps. Its interaction structure is defined by the (left) multiplicative behavior of elements in the semigroup Σ.[6] More precisely, we choose Σ as the set of cells as well as the set of input maps with interaction defined by multiplication from the left – τ receives an input of color σ from $\sigma\tau$. Performing the same construction for this network results in the same network again. This coins the name *fundamental network* denoted by $\widetilde{\mathcal{N}}$. The fundamental network of the running example can be seen in Figure 2.9. The arrow colors were deliberately chosen to be the same as in Figure 2.8 as they are determined by the same input maps.

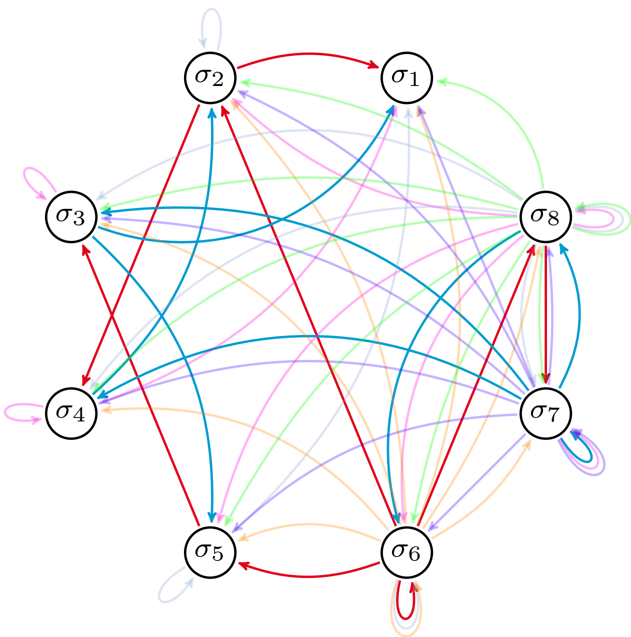


Figure 2.9: The fundamental network of the semigroup network $\mathcal{N}$ in Figure 2.8.[7]

Once again we have constructed a network that is more complicated than the one we started with. But also once more we gain additional understanding of network structure in return. As a matter of fact, the fundamental network has several properties that make it an object worth studying. First and foremost, besides the interactions being defined by the same maps, there is an actual relation between the semigroup network and its fundamental network with implications on dynamics. In particular, the interaction structure allows to attach the same internal phase space V to the cells of the fundamental network and let their internal dynamics be driven by the same function f. We denote the admissible vector fields of the fundamental network by Γ_f and call them *fundamental network vector fields*.

Theorem 2.23 (Theorem 10.1 in [92]). *Denote the total phase spaces of the semigroup network and its fundamental network by V^N and V^n respectively. Then the maps*

$$\varphi_p^*\colon V^N \to V^n, \quad (x_1, \dots, x_N) \mapsto (x_{\sigma_1(p)}, \dots, x_{\sigma_n(p)})$$

[6] It is constructed as the right Caley graph of Σ.

[7] Figure 2.9 is essentially Figure 4.1 in Schwenker [97].

conjugate γ_f to Γ_f, i.e.

$$\Gamma_f \circ \varphi_p^* = \varphi_p^* \circ \gamma_f$$

for all $p \in C$.

Similar to Section 2.4, this semiconjugacy implies, among other things, that solutions of a network dynamical system for the semigroup network are mapped to solutions of the corresponding fundamental network by the maps π_p. But also the other direction is possible, solutions of the semigroup network can be recovered from solutions of the fundamental network with slightly more effort. Another interesting observation is, that each of these semiconjugacies φ_p^* gives rise to a robust synchrony pattern of the fundamental network, where cells σ and τ are synchronous if $\sigma(p) = \tau(p)$ – i.e. $\mathrm{im}\left(\varphi_p^*\right)$ is a robust polydiagonal.

Even though the fundamental network has a more complex interaction structure, studying Γ_f instead of γ_f has certain advantages. Most importantly, the fundamental network vector fields are entirely characterized by symmetry. At this point we state the following theorem only informally.

Theorem 2.24 (Theorem 3.11 in [91]). *(i) There is a* representation *of the semigroup via linear maps on the total phase space defined by the multiplicative behavior of its elements:*

$$\sigma \to A_\sigma \in \mathfrak{gl}(V^n), \quad A_\sigma \circ A_\tau = A_{\sigma\tau}.$$

(ii) Every fundamental network vector field commutes with this representation

$$\Gamma_f \circ A_\sigma = A_\sigma \circ \Gamma_f, \quad \textit{for all} \quad \sigma \in \Sigma. \tag{2.9}$$

(iii) Every equivariant function is a fundamental network vector field.

In the first publication RINK and SANDERS [92] only the first symmetry statement (ii) was shown, which already has a major impact as it introduces symmetry into the fundamental network vector fields. These symmetries are not symmetries of the network in the classical sense. The fundamental network in general does not exhibit any non-trivial symmetries and the representation maps A_σ are not invertible. However, the *equivariance condition* (2.9) imposes strict restrictions on suitable vector fields just as in the classical equivariant dynamics theory. Furthermore, the fundamental network structure can equally be encoded in terms of this symmetry. Combining this result with Theorem 2.23, we see that these symmetries furthermore have an impact on dynamics of the semigroup network as well due to the semiconjugacy of the admissible vector fields (see also the discussion after Theorem 2.23). Hence, we refer to them as *hidden symmetries* of the original network $\mathcal{N}$. Since this discovery, several publications have been devoted to the adaptation of methods from equivariant dynamics to the semigroup context. Multiple results were derived such as an equivariant Lyapunov-Schmidt reduction. This provided the tools for a classification of steady state bifurcations in all 2- and 3-cell fundamental networks (RINK and SANDERS [91]). Newer results describe center manifolds (NIJHOLT, RINK, and SANDERS [79, 81]) as well as spectral properties in bifurcation problems in terms of networks and in terms of equivariance (NIJHOLT and RINK [77], NIJHOLT, RINK, and SANDERS [80], and SCHWENKER [97]).

Even though, the fundamental network construction proved to be powerful for the investigation of dynamics of semigroup networks, it required the introduction of graph fibrations (compare to Section 2.4) to thoroughly clarify the reasons for its existence, its properties, and its relation to the original networks. In NIJHOLT, RINK, and SANDERS [78] the authors provide a reformulation of their theory in terms of graph fibrations. It turns out that the semiconjugacy between the fundamental network and the original network (Theorem 2.23) as well as the symmetries of the fundamental network (Theorem 2.24) are instances of semiconjugacies that are induced by graph fibrations (see

Theorem 2.20). There are semiconjugacies φ_p^* between the fundamental network and the semigroup network for every cell $p \in C$ (Theorem 2.23). They are induced by graph fibrations of the form

$$\varphi_p \colon \Sigma \to C, \quad \sigma \mapsto \sigma(p)$$

which map the fundamental network to the original network (see Theorem 6.4 in NIJHOLT, RINK, and SANDERS [78]). On the other hand, the symmetries of the fundamental network are induced by so-called *self-fibrations*, i.e. graph fibrations of the fundamental network to itself. These are of the form

$$\varphi_\sigma \colon \Sigma \to \Sigma, \quad \tau \mapsto \tau\sigma$$

for all $\sigma \in \Sigma$ (see Theorem 7.2 in NIJHOLT, RINK, and SANDERS [78]). In both cases it is important to bear in mind, that the cells of the fundamental network are given by Σ. Note that the semiconjugacy condition (2.5) in Theorem 2.20 in the previous section contains the derivative of the induced map φ_p^*. However, in the case of vector spaces as internal phase spaces this is no longer necessary, since the induced map is linear .

This is an explicit example of the impact of the general tool of graph fibrations on network dynamics as they were already anticipated in DEVILLE and LERMAN [24, 25, 26]. Two rather simple types of graph fibrations allow us to introduce (or rather adapt) well-established machinery from equivariant dynamics to the analysis of networks. This nicely explains some of the similarities of phenomena observed in both areas like synchronous states or spectral degeneracies by hidden symmetries. Furthermore, it connects the fundamental network structure and its meaning for semigroup networks to the groupoid formalism. As we mentioned in Section 2.4, graph fibrations can be seen as a generalization of quotients by balanced colorings. Accordingly, we obtain semigroup networks as generalized quotients of their fundamental networks. We end this section with a small illustrative example for this idea and postpone any further discussion on semigroup and fundamental networks to the main body of this thesis.

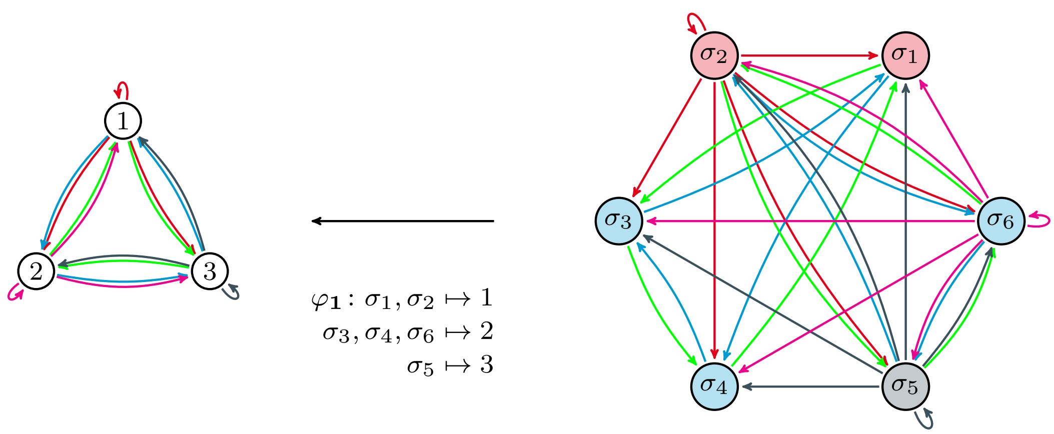


Figure 2.10: A 3-cell semigroup network that is a quotient by a balanced coloring of its fundamental network.

Example 2.8. For a cell that receives inputs from every other cell in the network, the graph fibration φ_p is surjective. As a result the image of the induced semiconjugacy is a robust synchrony subspace of the fundamental network. The graph fibration is then the same as the quotient map by a balanced coloring in the groupoid formalism. Hence, the semigroup network is contained in the fundamental network as a quotient and its dynamics is characterized by the restriction of the fundamental network vector fields to this polydiagonal. Consider the semigroup network on the left hand side and its fundamental network on the right hand side of Figure 2.10. Cell 1 receives an input from any other cell in the network. The corresponding graph fibration $\varphi_{\mathbf{1}}$ maps σ_1 and σ_2 to 1, σ_3, σ_4, and σ_6

to 2 and σ_5 to 3. As this is clearly surjective, it is in fact a quotient map that identifies the cells in the fundamental network with the same image. The corresponding coloring can be seen in the right hand side of Figure 2.10. △

2.6 Asynchronous networks

The remaining two sections of this part introduce two very recent approaches to network dynamics that have a similar driving idea – namely *modularity* – but employ completely different techniques. As these are, furthermore, quite distinct from (Section 2.6) or geared towards less specialized dynamics (Section 2.7) than the formalism we work with in this thesis, we omit most of the details and summarize only the main ideas. The concept of modularity, as we have mentioned before, describes the observation that a network is composed of parts – e.g. nodes, subnetworks, quotients –, often referred to as *building blocks*, to form a whole (the same holds for other structures as well). It is desirable to deduce understanding about the dynamics of the entire network from the parts building it.

In BICK and FIELD [19, 20, 21] the authors approach this concept from an engineering perspective, by focusing on the *function* of a dynamical network. That is, given some *initialization*, what *termination* state does the network reach after some finite time. They realized that many real world problems, even though exhibiting structural properties of a network, cannot be modeled by smooth dynamical systems on fixed networks as is common in the area of network dynamics. Think for example of a parallel computer. Therein, multiple computations are carried out simultaneously, e.g. by cores that are the nodes, and then processed jointly once all computations are terminated. Hence, interaction happens only in specific states and the internal dynamics varies accordingly, i.e. the task carried out by each node during interaction is different from the computation before. Furthermore, restricted dynamics is necessary. Cells have to wait for others to have completed a certain process before communication or execution of their own tasks can continue. As a matter of fact, cells in general do not even share the same time frame. For example, two carriers can only exchange goods when they are located in the same storage facility. This is independent of starting time, speed, and route before joining each other at a common location. In order to model such applications, one needs a framework in which each cell has its own time frame, the interaction structure between cells varies throughout the process as do the internal dynamics, and restricted dynamics, such as waiting, needs to be possible. These requirements lead to an extension of the network dynamics theory to non-smooth dynamics.

The authors develop a theory of *asynchronous networks* and *event driven* dynamics. The term *asynchronous* refers to the fact that each cell has its own time frame. Nevertheless, two cells may still synchronize in the classical sense, if they happen to follow the same 'clock'. The dynamics is *event driven* due to the changes in structure as well as in governing functions at certain events. We summarize the definitions informally.

Definition 2.25 (Definition 4.12 and Section 4.6 in [20]). *An* asynchronous network *consists of*

(i) a set of cells each with an internal phase space, i.e. a finite-dimensional vector space or differentiable manifold,

(ii) a collection of interaction structures – there can either be one directed connection or no directed connection from one cell to another and no self loops –,

(iii) an admissible vector field for each of the interaction structures, for example in the sense of the groupoid formalism,

(iv) an event map sending each point in the total phase space to one of the interaction structures.

Restricted dynamics, such as stopping, are governed by one specific constraining *node. When there is a connection to another cell, its dynamics is constrained according to the specific* constraint structure. *The admissible vector fields are then given by the interaction structure and the corresponding admissible vector field determined by the value of the event map.*

Non-smoothness is introduced into the model by the event map. It picks an interaction structure as well as a dynamical system for each state of the network. For a given event, however, everything is defined according to the known formalisms. The constraint structure can then be used to impose certain restrictions, e.g. algebraic ones, on the dynamics of a node when it is constrained at a given event. This induces certain regularity conditions on the possible admissible vector fields in order to guarantee well-definedness. Furthermore, one may include so-called *local times* into the framework. To that end, each strongly connected (i.e. all-to-all coupled) component of the network obtains a local time. This is driving dynamics according to the admissible vector fields and propagated at a constant rate that differs throughout the components. As the components vary with the events, so do the local times. As a result, two cells synchronize their local time, when they are connected at a given event. Even though the dynamical systems governed by these types of admissible vector fields are non-smooth and highly complex, under certain regularity conditions – in particular concerning the event map – they induce semiflows of solutions. That is, each initial condition of the network can be propagated in forward time by a piecewise smooth (with respect to time) function (Proposition 4.25 in Bick and Field [20]).

A large part of the theory revolves around the study of these semiflows in terms of *functionality*. To that end, one investigates finite-time dynamics, more precisely the question whether the flows starting in certain *initializing sets* reach given *termination sets* in finite time and, if so, in which points and what time is required for the transition. An asynchronous network together with initializing and termination sets is called a *functional asynchronous network*. Its *function* is transitioning any initialization state to the corresponding termination state.

Obviously, not all combinations of these sets are possible. However, a functional network with suitable initialization and termination sets can be interpreted as a black box fulfilling its function. As a result, one can build larger functional networks by feeding the termination of one network into the initialization of another. Other combination methods, such as parallel networks, are possible as well. Hence, we obtain a modular structure of functional networks in terms of building blocks with input/output-relations given by their function. The most important feature of this building method is that the transmission in one building block occurs in forward flowing time. Hence, complicated combinations such as feedback-loops are prohibited. The authors derive a powerful result on the dynamics of a functional asynchronous network that captures the spirit of the main question of modularity: the factorization and modularization of dynamics theorems.

Theorem 2.26 (compare to Theorem 1 in [19])**.** *Under general conditions, a functional asynchronous network has a unique (up to rearrangement) decomposition into building blocks. Its function can be expressed in terms of the function of its building blocks.*

This theorem provides deep understanding of modularized dynamics. However, certain restrictions still need to be accepted. In particular, the theorem does not provide general knowledge about the *dynamics* of the network from understanding of all its building blocks. On the other hand, not many results regarding modularity of dynamics in that generality are known. This result is an essential step in that direction, as it allows to deduce the networks *function* precisely from that of its building blocks, i.e. the transition of initializing states into termination states. If one agrees, that the function of a network is its key feature – which is certainly the case in the applications presented in the references –, this result provides all the important information via a modular computation. Furthermore, it allows for great benefits in the design of networks with a predefined functions which could prove to be very useful in suitable applications. All the details on the constructions of networks and their function as well as several examples, including applications of trains on single lane tracks that need to pass each other in certain stations, can be found in the mentioned references that we recommend for further reading.

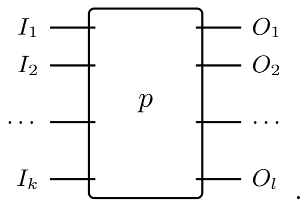
2.7 Modular systems and operad algebras

The final approach we introduce focuses strongly on the concept of *modularity* by introducing category, operad, and sheaf-theory into network dynamics. This abstractly algebraic theory bears resemblance to the one developed by DeVille and Lerman (Section 2.4). Once again, we try to keep the presentation as simple as possible in order to transport only the main ideas. To that end, we use the language of category theory without precise definitions and *highlight* new terms on first occasion. For a more in-depth introduction, the reader should consult the mentioned references.

The main idea in the theory is to consider networks as *systems of systems* in the sense that a network consists of smaller parts that are connected, each of which can be considered as a system in its own right – the authors use the keyword *compositionality* instead of modularity. These parts can be single units or nodes of the network but also entire subnetworks. This idea can be thought of as *zooming* into a given network to arbitrary depth. This formalism shares its philosophy with asynchronous networks (Section 2.6). Contrary to the notion therein in which interaction of agents requires the flow of time – with good reasons from the applications in mind – here communication between building blocks in general happens instantaneously. This is also implicit in the other approaches we mentioned.

The theory was developed in a series of papers by Spivak and collaborators (Rupel and Spivak [93], Spivak [101, 102], Spivak, Schultz, and Rupel [103], and Vagner, Spivak, and Lerman [112]) that is summarized and further extended using sheaf theory in Schultz, Spivak, and Vasilakopoulou [96]. Parts of it are also presented from a broader perspective in the beautiful book on applied category theory Fong and Spivak [39]. We also recommend this reference for background on the category theoretic aspects of this section. Furthermore an extension to networks with time-varying topology, called *mode dependent networks* by the authors is, is available (see Spivak and Tan [104]). The approach has since been successfully applied to model an Aircraft Collision Avoidance System (ACAS) inspired by a real-world system that is used in airplanes to prevent in-air collisions (see Speranzon, Spivak, and Varadarajan [100]).

The corner stone of the formalism is the *symmetric monoidal category* $(\mathcal{W}_\mathcal{C}, \oplus, 0)$ that is called the category of *$\mathcal{C}$-labeled boxes and wiring diagrams* (see Definition 2.2.2 in Schultz, Spivak, and Vasilakopoulou [96]). The objects of this category are 'empty' boxes with collections of input and output ports $p = (I, O) = (\{I_1, \dots, I_k\}, \{O_1, \dots, O_l\})$



Each port is assigned with an element of the *typing category* $\mathcal{C}$ that determines what type of information gets passed through a given port into or out of the box X.

These boxes can be connected – or *wired* – in two different ways to build networks. *Parallel composition* turns two boxes into a larger one with the inputs and outputs of both individual boxes combined. This makes for the monoidal structure $\oplus$. Secondly, boxes can be nested using the morphisms or maps in the category $\mathcal{W}_\mathcal{C}$. A map $\varphi\colon p \to q$ in the category is called a *wiring diagram* as it describes how the box p sits in the box q, i.e. how the inputs of p are connected to the inputs of q as well as the outputs of p, and how the outputs of p are connected to the outputs of q. Iteratively combining these methods allows to connect boxes to form arbitrary networks, which completes the algebraic framework. An example of this construction method is shown in Figure 2.11. The next step is to construct dynamical objects that can fill or inhabit the boxes in a way that the algebraic operations for boxes are respected by the dynamics of its inhabitants. On a technical note, the construction involves identifying the category $\mathcal{W}_\mathcal{C}$ with its underlying *symmetric colored operad* to encode the wiring of N cells in terms of N-ary morphisms using the monoidal structure. This identification provides the technical background in other parts of the formalism as well. However, we do not include any more details in this text.

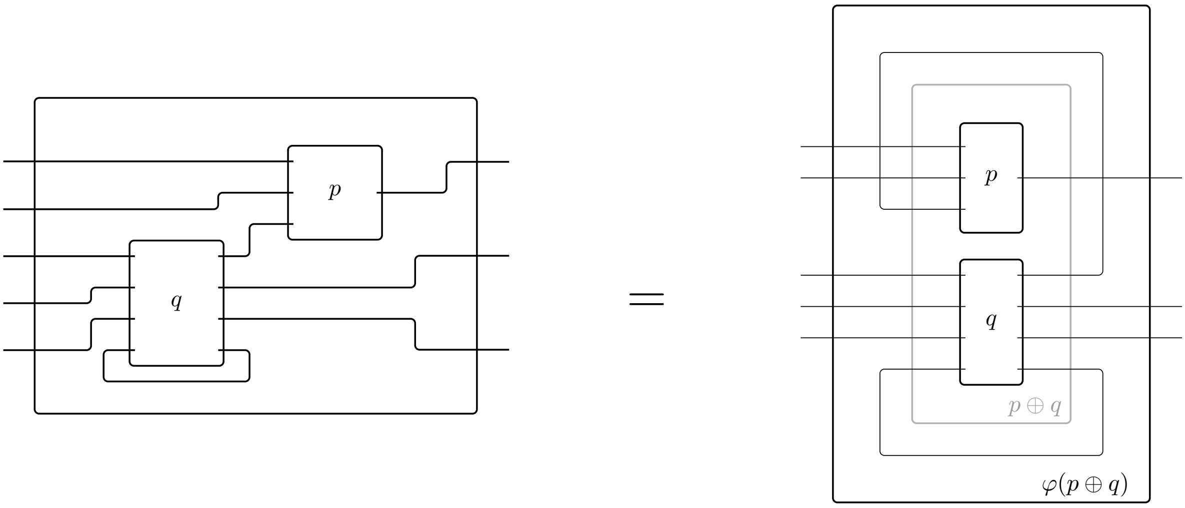


Figure 2.11: A wiring of two boxes to build a network.

In order to incorporate dynamics into networks described by wired boxes the authors widely generalize the notion of dynamical systems. They propose objects describing dynamical behavior in different time frames to inhabit boxes and exchange inputs and outputs according to wiring diagrams. As we have mentioned in the presentation of asynchronous networks, this yields severe problems, especially when modeling real world systems. In particular, the construction requires to begin by translating the different notions of time into one, that is suitable for all of the subsystems. A natural way of doing so comes with the description of time dependent behavior in terms of so-called *(continuous) interval sheaves* (see Definition 3.2.2 in SCHULTZ, SPIVAK, and VASILAKOPOULOU [96]). These include a common notion of time for subsystems that interact. Informally speaking, interval sheaves are functions, mapping intervals into a given set, that can uniquely be glued together. This property allows for the interconnection of different subsystems. The intervals can then be interpreted as a notion of time and the sheaf describes flow of information or data, in order to model evolving processes. We use sheaves to describe states, inputs, and outputs of boxes. Due to their flexibility, this formalism introduces very general types of dynamics – various types of information flows can be defined in this language. Interval sheaves together with translations form a category that we denote by $\widetilde{\textbf{Int}}$. We omit any further details at this point and refer to SCHULTZ, SPIVAK, and VASILAKOPOULOU [96] for further information.

In order to include interval sheaves into the network formalism, we let them inhabit boxes in wiring diagrams. That is, to a box $p = (I, O)$ we assign three sheaves $\mathcal{S}, \mathcal{S}^{\text{in}}$ and $\mathcal{S}^{\text{out}}$ together with a *sheaf map* $s = (s^{\text{in}}, s^{\text{out}})\colon \mathcal{S} \to \mathcal{S}^{\text{in}} \times \mathcal{S}^{\text{out}}$ – i.e. a *span* of the input and output sheaves. These sheaves describe the evolution of the state, the input, and the output respectively. The relation between input and output is determined by the state sheaf $\mathcal{S}$ and the map s. We refer to these objects as *(continuous) machines* (see Definition 4.1.1 in SCHULTZ, SPIVAK, and VASILAKOPOULOU [96]). According to the notation from before, the category of boxes inhabited by machines is denoted by $\mathcal{W}_{\widetilde{\textbf{Int}}}$.

Interval sheaves provide sufficient flexibility to allow for arbitrary wiring, composition and nesting of machines according to the rules defined for boxes and wiring diagrams. The operations on boxes and wiring diagrams induce similar ones on machines. One obtains maps transforming state, input and output sheaves into new ones for the interconnected systems. This provides a new machine encompassing the behavior of all of its components. This new machine guarantees that interacting components follow the same time frame. Formally speaking, continuous machines form an *algebra for the symmetric colored operad of wiring diagrams* $\mathcal{W}_{\widetilde{\mathbf{Int}}}$ (see Proposition 4.1.3 in SCHULTZ, SPIVAK, and VASILAKOPOULOU [96]).

Certainly, in this generality problems regarding consistency of a network of continuous machines may arise. In SCHULTZ, SPIVAK, and VASILAKOPOULOU [96] the authors, provide conditions on the sheaf map s to guarantee *totality* and *determinism* of interconnected systems (see Proposition 4.2.1 and Definition 4.2.2 in SCHULTZ, SPIVAK, and VASILAKOPOULOU [96]). These properties denote the persistence of a machines state for a given input/output-configuration when time – and hence also input – flows over a short period of time. The terms correspond to existence and uniqueness properties in the theory of ordinary differential equations which is a sufficient way of thinking about them for the scope of this text. The mentioned conditions are independent of the specific wiring. They impose restrictions on the sheaves that can be attached to boxes. Hence, they allow only for a restricted class of dynamics but keep all possible interconnections admissible.

One type of dynamics that can be described in terms of interval sheaves are so-called *open systems* (see Definition 2.3.3 in SCHULTZ, SPIVAK, and VASILAKOPOULOU [96]). This formalism from control theory is used to encode dynamics that is in interaction with its environment similar in spirit to the control systems from Section 2.4 (Definition 2.17) but slightly more general. The time evolution of the state variable $x \in V$ depends on some (time dependent) input $x^{\mathsf{in}} \in V^{\mathsf{in}}$. Furthermore, the system produces an output $x^{\mathsf{out}} \in V^{\mathsf{out}}$. We only employ internal dynamics in finite dimensional vector spaces here. However, a generalization to manifolds is also available. The system is denoted by $F = (V, f^{\mathsf{dyn}}, f^{\mathsf{rdt}})$ with smooth dynamics and readout functions

$$\begin{cases} f^{\mathsf{dyn}} & : V^{\mathsf{in}} \times V \to V \\ f^{\mathsf{rdt}} & : V \to V^{\mathsf{out}}. \end{cases} \tag{2.10}$$

It is to be interpreted as a vector field in V that depends on parameters in V^{in} and furthermore produces an output in V^{out}. This reflects the idea of *black boxes*, of which we only know and care about the input/output relation but not about the internal state. Such open systems offer a class of dynamical systems that can be interconnected in a natural way. The output of one system can be plugged into the input of a second one as long as their input and output spaces coincide. This allows to create networks of systems that can be of diverse types. Especially, their state spaces do not interfere with coupling. All that matters is the compatibility of input and outputs of individual systems.

Then we may assign the smooth machine $\mathcal{S}^{\mathsf{in}} \xleftarrow{s^{\mathsf{in}}} \mathcal{S} \xrightarrow{s^{\mathsf{out}}} \mathcal{S}^{\mathsf{out}}$ to the open dynamical system $(V, f^{\mathsf{dyn}}, f^{\mathsf{rdt}})$. The sheaf $\mathcal{S}$ describes the smooth trajectories and also encompasses the inputs. It is defined via sections given by

$$\mathcal{S}(\ell) = \left\{ (x^{\mathsf{in}}, x) \colon [0,\ell] \to V^{\mathsf{in}} \times V \;\middle|\; x^{\mathsf{in}}, x \text{ are smooth and } \frac{dx}{dt} = f^{\mathsf{dyn}}(x^{\mathsf{in}}, x) \right\}.$$

The sheaf maps s^{in} and s^{out} map the section onto the corresponding inputs and outputs via the output function f^{rdt}:

$$\begin{aligned} s^{\mathsf{in}}(x^{\mathsf{in}}, x) &= x^{\mathsf{in}} \\ s^{\mathsf{out}}(x^{\mathsf{in}}, x) &= f^{\mathsf{rdt}}(x). \end{aligned}$$

Their images are the input and output sheaves $\mathcal{S}^{\mathsf{in}}$ and $\mathcal{S}^{\mathsf{out}}$ (see Proposition 5.1.2 in SCHULTZ, SPIVAK, and VASILAKOPOULOU [96]).

Summarizing, we can define a highly general class of time dependent systems that allows for arbitrary interconnections and nesting. Via construction all subsystems share a common conception of time when they interact which increases constructability. The systems can describe dynamics induced by ordinary differential equations. But also much more general types of dynamical systems, such as systems with discrete states such as piecewise constant signals or hybrid versions can be defined in a similar manner. This flexibility makes for a great potential in real world applications. We can model highly diverse types of dynamics in which information or data passes through an underlying network structure, such as power grids, transport networks, neuronal networks and many others. Compositionality, furthermore, allows to investigate such networks from different scales as it allows to 'zoom' in and out of the structure.

Analytically, this comes with its up- and downsides. The formalism – in particular the algebra of continuous machines for the operad of wiring diagrams – has already proven to be powerful for a modular analysis of network dynamics. That means, we can analyze a full system made up of smaller building parts by using knowledge of its parts. The methods of computation are in accordance to the rules of wiring of building parts. For example in SPIVAK [102], the authors investigate steady states of interconnected systems as an example. They show that these can be encoded in a generalized matrix. The *steady state matrix* of an interconnected system can then be computed from those of its components using basic matrix arithmetics. The rules of computation are compatible with the operations of wiring of systems and remain the same for all possible systems. The authors also provide a short discussion of computational savings from the use of steady state matrices.

On the other hand, the flexibility of the approach potentially leads to the presence of various different types of data that need to be taken into account in computations. This in turn, makes analytical investigations of dynamics much more complicated. Furthermore, the compositional computations are not necessarily simpler than classical techniques. So far, no notion of genericity has been introduced into this formalism. One would like to identify implications that a given network structure has on any – or at least almost any in some sense – dynamical system with the network as its underlying structure. This seems to be a difficult task in the present formalism. It is possible to attach very different kinds of dynamical systems to networks of boxes and wiring diagrams. Therefore, it is even difficult to coin a common conception of genericity that can be applied to these diverse dynamics.

Remark 2.27. The formalism presented in this section was developed in parts simultaneously to the category theory inspired approach to modular dynamics using graph fibrations that we presented in Section 2.4. It is similar in spirit even though the focus is different. In DEVILLE and LERMAN [25] the authors refer to VAGNER, SPIVAK, and LERMAN [112] by mentioning the fact that "Open continuous time systems form an algebra over a certain operad", which they identify in their work as well as implicitly in the combinatorial approach to networks (Section 2.3). They claim that the relation between this operad and graph fibrations is not fully understood and postpone a thorough investigation to future work. Recently, attempts have been made to close this gap. In LERMAN [72] (which is a follow-up of the preprint LERMAN and SPIVAK [73]) the idea of interpreting a network in terms of a colored operad is used to generalize the framework of networks of control systems previously used in the approach by DEVILLE and LERMAN. This allows for a natural way of building larger networks from existing ones – the networks in Section 2.4 are fixed and modularity is restricted to the investigation of subsystems or quotients. Furthermore, a generalized notion of maps between networks of open systems is proposed. This allows for generalized synchronization results similar to those obtained by graph fibrations and quotient maps. As a matter of fact, in Section 10 of LERMAN [72], the author thoroughly discusses how this theory extends the definitions and results from DEVILLE and LERMAN [24, 25, 26]. On the other hand, it provides new operadic approaches to networks that allow for further understanding of modularity (in particular of maps of networks), that were not possible in the setting presented in this section. The general forms of dynamics described in the language of sheaves, however, is not pursued. $\triangle$

Part II

Genericity in homogeneous coupled cell systems

Contents

Chapter 3

Hidden symmetry in semigroup networks

This chapter marks the beginning of the main body of this thesis which stretches over Parts II and III. Our results further develop the formalism of semigroup network dynamical systems – to which we mostly refer as homogeneous coupled cell systems, homogeneous coupled cell networks, or simply as networks. In Part II we present the main results regarding bifurcations in parameter dependent systems. We mainly exploit the hidden symmetries introduced in Section 2.5 to use and adapt techniques from equivariant bifurcation theory. This chapter sets up the underlying formalism more thoroughly than it was done in the introduction. It was mainly developed in NIJHOLT, RINK, and SANDERS [78] and RINK and SANDERS [91, 92] with some additions given in NOETHEN [85]. Therein one can also find adaptions of the established techniques of normal forms [92] and Lyapunov-Schmidt reduction [91]. A network version of the center manifold reduction was later added in NIJHOLT, RINK, and SANDERS [79, 81]. Additionally, AGUIAR, DIAS, and SOARES [10] investigate properties of the fundamental network and NIJHOLT, RINK, and SANDERS [80] describe the impact of certain substructures of a network on its dynamics. Finally, we would like to mention two pieces of work that focus on the theoretical aspects of equivariant dynamics with respect to generalized symmetry: NIJHOLT and RINK [77] and [97] by the author of this text. These results are also part of this thesis and will be presented in the upcoming chapters. Some overlap with Section 2.5 cannot be avoided. However, we provide more details at this point. On the other hand, we omit any motivation and historical background and refer to the introductory section for these information. Our work on semigroup networks has also led to some additional, mostly technical, results or reformulations that are not part of the aforementioned references. We chose to include some of these at points were they fit into the context at the risk of inducing confusion which results are novel. We state results that were not provided by the author of this thesis and his co-authors without proofs. Furthermore, we indicate these definitions and statements with references to the literature.

3.1 Semigroup networks, fundamental networks, and hidden symmetry

3.1.1 Homogeneous coupled cell systems with asymmetric inputs

We are interested in *homogeneous networks with asymmetric inputs*, i.e. networks in which all cells are of the same type and receive precisely one input of each type. In a graphical interpretation all nodes have the same shape but all arrows pointing at one cell have different colors. The interaction structure can be encoded in terms of functions from the set of cells to itself.

Definition 3.1 (Section 2 in [92])**.** *A* homogeneous (coupled cell) network with asymmetric inputs $\mathcal{N}$ *consists of a finite set of* cells *or* nodes C *with* $\#C = N$ *and a finite set of* input maps Σ *with* $\#\Sigma = n$.[1] *Each input map* $\Sigma \ni \sigma\colon C \to C$ *can be interpreted as one arrow color, i.e. cell* $p \in C$ *receives an input of color* σ *from cell* $\sigma(p)$.

[1]The symbol $\#$ denotes the number of elements in a finite set.

Note that this notation implicitly requires the identifications of arrow colors that have the same interaction structure. That is, if $\sigma' = \tilde{\sigma}$ are the same input maps, the corresponding arrow colors provide the same inputs for each cell. Hence, we interpret them as only one arrow color σ. The notation of interactions in terms of maps allows for a convenient definition of paths in a network.

Definition 3.2. *A* (directed) path from p to q *for two cells* $p, q \in C$ *is the sequence* $\omega_{p,q} = (p_1, \dots, p_k)$*, where* $p_1 = p, p_k = q$ *and there exist* $\sigma_1, \dots, \sigma_{k-1} \in \Sigma$ *such that* $\sigma_1(p_k) = p_{k-1}, \dots, \sigma_{k-1}(p_2) = p_1$*. If all* p_i *are pairwise different the path does not contain any loops. Denote the set of all paths without loops from* p *to* q *by* $\Omega_{p,q} = \{\omega_{p,q} \mid \omega_{p,q}$ *has no loops*$\}$.

Sometimes it is handy to look only at those parts of a network, that have an input into a given cell.

Definition 3.3 (Definition 6.5 in [78])**.** *Fix a cell* $p \in C$*. Its* input network $\mathcal{N}_p$ *has the set of cells*

$$C_p = \{p\} \cup \{q \in C \mid \Omega_{p,q} \neq \emptyset\}$$

and input maps Σ.

Note that any path into a cell in C_p is also a path into p so that the input network is well-defined. Contrary to the input tree in the groupoid formalism we do not artificially distinguish inputs according to arrow colors. A cell might have multiple inputs into p via different arrow colors. Furthermore, by including all input maps into the input network of p it also contains interactions between $q, q' \in C_p$ when $q, q' \neq p$.

In order to incorporate dynamics into these networks, we attach a finite-dimensional vector space V to each cell that we call the *internal phase space*. As the network is homogeneous, this is the same for all cells. The *total phase space*, i.e. the phase space of the entire network, is the direct sum $\bigoplus_{p\in C} V \cong V^N$. To reflect the networks cells we choose coordinates as $x = (x_p)_{p\in C} = (x_{p_1}, \dots, x_{p_N})$ (the last representation is only reasonable when an ordering of the cells has been chosen). As a cell receives an input via every input map, the *coupling phase space* (in the language of the groupoid formalism) is $\bigoplus_{\sigma\in\Sigma} V \cong V^n$. Note that this is not the same as the total phase space of the input network of a cell. Furthermore, the internal dynamics is governed by the same function for each cell.

Definition 3.4 (Definition 2.1 in [92])**.** *Let* $f \in \mathcal{C}^\infty(\bigoplus_{\sigma\in\Sigma} V, V)$ *be a smooth* response function *– also referred to as* internal dynamics*. Then the* homogeneous coupled cell vector field *– or* admissible vector field *– is defined by*

$$\gamma_f(x) = \begin{pmatrix} f(x_{\sigma_1(p_1)}, \dots, x_{\sigma_n(p_1)}) \\ f(x_{\sigma_1(p_2)}, \dots, x_{\sigma_n(p_2)}) \\ \vdots \\ f(x_{\sigma_1(p_N)}, \dots, x_{\sigma_n(p_N)}) \end{pmatrix}, \tag{3.1}$$

where we have assumed an ordering of the cells and the input maps. A homogeneous coupled cell system *is a system of ordinary equations governed by an admissible vector field*

$$\dot{x} = \gamma_f(x).$$

Remark 3.5. We focus exclusively on smooth time-continuous dynamics in this thesis. However, the formalism can readily be adapted to describe systems with less regularity as well as discrete dynamics. △

3.1.2 Relations to other formalisms

We have briefly introduced some general concepts for the investigation of network dynamics in Chapter 2. It turns out, that the formalism of encoding interactions via input maps greatly simplifies some of these notations and results. For example, one often describes the network structure in terms of adjacency matrices. A general notion is that of defining an $N \times N$-matrix for every type of input such that the (i, j)-th entry is the number of arrows of that type from cell p_j to p_i. Here we use a slightly more general concept by extending this definition to the total phase space.

Definition 3.6. *For each $\sigma \in \Sigma$ define a linear map $B_\sigma \colon \bigoplus_{p\in C} V \to \bigoplus_{p\in C} V$ by $(B_\sigma(x))_p = x_{\sigma(p)}$. Choosing an ordering of the cells and coordinates according to the cells as before, this is an $N \times N$ matrix with entries from the linear maps on the internal phase space V. We refer to the B_σ as the* adjacency matrix of the arrow color *σ.*

Remark 3.7. Choosing a basis for V, we may interpret these matrices as $N \times N$ block matrices. Their structure is independent of the internal phase space:

$$(B_\sigma)_{p,q} = \begin{cases} \mathbb{1}_V, & \text{if} \quad \sigma(p) = q \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathbb{1}_V, 0 \in \mathfrak{gl}(V)$ are the identity and the 0-map respectively. Here $\mathfrak{gl}(V)$ denotes the algebra of linear maps on V. In particular, if the internal phase space is $V = \mathbb{R}$, we may identify $\mathfrak{gl}(\mathbb{R})$ with $\mathbb{R}$ so that the entries are either 1 or 0. In that case, we obtain the classical adjacency matrices. △

Usually, we do not make use of the fact that adjacency matrices encode the interaction structure precisely. However, extending them to linear maps on the total phase space has interesting implications for linear network vector fields. In particular, the adjacency matrices span the space of linear admissible maps in the sense that is made precise in the following proposition.

Proposition 3.8. *Let $L \colon \bigoplus_{p\in C} V \to \bigoplus_{p\in C} V$ be linear and admissible, i.e. defined as in* (3.1) *for some linear response function. Then L is of the form*

$$(Lx)_p = \sum_{\sigma\in\Sigma} b_\sigma (B_\sigma(x))_p \tag{3.2}$$

where $b_\sigma \in \mathfrak{gl}(V)$ are linear maps on V independent of p.

Proof. A linear admissible map L is uniquely defined by a linear response function $\mathfrak{l} \colon \bigoplus_{\sigma\in\Sigma} V \to V$, i.e. $L = \gamma_{\mathfrak{l}}$. As $\mathfrak{l}$ is linear and its arguments are labeled by the input maps $\sigma \in \Sigma$, we find $b_\sigma \in \mathfrak{gl}(V)$ such that

$$\mathfrak{l}(Y) = \sum_{\sigma\in\Sigma} b_\sigma Y_\sigma,$$

where $Y = (Y_\sigma)_{\sigma\in\Sigma} \in \bigoplus_{\sigma\in\Sigma} V$. Then for $x \in \bigoplus_{p\in C} V$ the p-th entry of Lx depends on the entries x_q such that there is an arrow from q to p, i.e. there is $\sigma \in \Sigma$ such that $\sigma(p) = q$. We obtain

$$(Lx)_p = \mathfrak{l}(x_{\sigma_1(p)}, \ldots, x_{\sigma_n(p)}) = \sum_{\sigma\in\Sigma} b_\sigma x_{\sigma(p)} = \sum_{\sigma\in\Sigma} b_\sigma (B_\sigma(x))_p.$$

□

Corollary 3.9. *If $V = \mathbb{R}$ any linear admissible map L is a linear combination of the B_σ, i.e.*

$$L = \sum_{\sigma\in\Sigma} b_\sigma B_\sigma$$

for some $b_\sigma \in \mathbb{R}$ using the identification $\mathfrak{gl}(\mathbb{R}) \cong \mathbb{R}$.

As we mentioned in Section 2.1, symmetric networks receive a considerable amount of interest. Broadly speaking, a symmetry of a network is a permutation of its cells that leaves the interaction structure intact. In particular, the cells that have an input into cell $p \in C$ get mapped to cells that have an input into the image of p via an arrow of the same color. In terms of graph theory a symmetry is an automorphism of the directed multigraph that depicts the network structure. Such symmetries can readily be expressed in terms of input maps (Section 7 in Rink and Sanders [92]).

Definition 3.10 (p. 3523 in [92]). *A* symmetry *of the network $\mathcal{N} = (C, \Sigma)$ is a permutation $\mathbf{p}\colon C \to C$ such that*

$$\mathbf{p} \circ \sigma = \sigma \circ \mathbf{p}$$

for all $\sigma \in \Sigma$. The set of all symmetries of a network is called its symmetry group. *A network with non-trivial symmetry group is called a* symmetric network.

The symmetry group obviously has the algebraic structure of a group. Note, furthermore, that the admissible vector fields of a symmetric network are equivariant with respect to the natural representation of the symmetry group on the total phase space. That is, the maps

$$\mathbf{P}_{\mathbf{p}}\colon \bigoplus_{p\in C} V \to \bigoplus_{p\in C} V, \quad (x_p)_{p\in C} \mapsto (x_{\mathbf{p}(p)})_{p\in C}$$

satisfy

$$\mathbf{P}_{\mathbf{p}} \circ \mathbf{P}_{\mathbf{p}'} = \mathbf{P}_{\mathbf{p}\circ\mathbf{p}'}$$

and

$$\gamma_f \circ \mathbf{P}_{\mathbf{p}} = \mathbf{P}_{\mathbf{p}} \circ \gamma_f \tag{3.3}$$

for all admissible vector fields. Hence, the dynamics of the network can be investigated by the likes of the well-established theory of equivariant dynamics (see for example CHOSSAT and LAUTERBACH [23], FIELD [36], and GOLUBITSKY, STEWART, and SCHAEFFER [58]).

The next aspect to notice is that the definition of homogeneous coupled cell networks and their admissible vector fields is in accordance with the groupoid formalism presented in Section 2.2 (see also Section 5 in NIJHOLT, RINK, and SANDERS [78]). In particular, there is precisely one input isomorphism between any two cells mapping p to q and any arrow pointing at p onto the unique arrow of the same color pointing at q. This reflects in the same internal phase space and the same response function for each cell. One of the most important features of network dynamics is (partial) synchrony. That is, a solution of the coupled cell system in which the state variables of certain cells are equal. Robust patterns of synchrony are those, that are invariant under the dynamics of all admissible vector fields. Recall that these are in one-to-one correspondence to balanced colorings (in the language of the groupoid formalism (Theorem 2.8)) or balanced partitions (in the combinatorial language). We do not restate the details here but rather point out that these results can be reformulated for homogeneous coupled cell networks in terms of input maps.

Proposition 3.11 (Proposition 7.2 in [92]). *Let $P = \{P_1, \dots, P_r\}$ be a partition of the set of cells C. Then the following are equivalent*

(i) For every $f \in \mathcal{C}^\infty(\bigoplus_{p\in C} V, V)$ the subspace

$$\Delta_P = \left\{ x \in \bigoplus_{p\in C} V \;\middle|\; x_p = x_q, \quad \textit{if} \quad p, q \in P_i \quad \textit{for some } 1 \le i \le r \right\}$$

is invariant under the dynamics of γ_f, i.e. it is a robust synchrony subspace.

(ii) For all $\sigma \in \Sigma$ and $1 \le i \le r$ there exists $1 \le j \le r$ such that $\sigma(P_i) \subset P_j$.

Hence, the condition in (ii) of that proposition determines all balanced partitions in the network.

Finally, we recall another tool for the investigation of dynamics and the influences of network structure thereon that are inspired by category theory, namely graph fibrations (see Section 2.4). These maps between networks respect the interaction structure and can be used to encode structural properties of a network. Examples include dependency relations, subnetworks, quotients, and balanced partitions. We do not restate the general definition here but rather a reformulation in terms of input maps. The more restrictive setting allows us to simplify notation compared to Section 2.4.

Definition 3.12 (Proposition 5.3 of [78]). *Let $\mathcal{N}_1 = (C_1, \Sigma_1)$ and $\mathcal{N}_2 = (C_2, \Sigma_2)$ be two networks. Then we call a map $\varphi\colon C_1 \to C_2$ a* graph fibration *$\varphi\colon \mathcal{N}_1 \to \mathcal{N}_2$ if and only if there is a bijection $\varphi^\Sigma\colon \Sigma_1 \to \Sigma_2$ such that for all input maps $\sigma \in \Sigma_1$ we have*

$$\varphi \circ \sigma = \varphi^\Sigma(\sigma) \circ \varphi.$$

Remark 3.13. In particular, a necessary condition for the existence of a graph fibration between two networks is that both have the same number of input maps. In graphical terms that means, we can assign the same set of colors to the arrows of both networks so that the bijection φ^Σ is the identification of arrow colors in different networks. △

Graph fibrations are of great importance as they do not only relate two networks but also their corresponding dynamics. We may assign the same internal phase spaces to any two homogeneous coupled cell networks. If they additionally have the same number of input maps – identified as in the previous remark – each cell receives the same number of inputs from the same vector space in both networks. Hence, we can consider dynamics governed by the same response function in both networks and compare them. A graph fibration induces a semiconjugacy between the corresponding admissible vector fields which strongly impacts their dynamics. We restate the following theorem to make this more precise.

Theorem 3.14 (Theorem 4.3 in [78][2]). *Let $\mathcal{N}_1 = (C_1, \Sigma_1)$ and $\mathcal{N}_2 = (C_2, \Sigma_2)$ be two networks and $\varphi\colon \mathcal{N}_1 \to \mathcal{N}_2$ a graph fibration. Then φ induces a linear map between the total phase spaces defined by*

$$\begin{aligned} \varphi^*\colon \bigoplus_{p' \in C_2} V &\to \bigoplus_{p \in C_1} V \\ (\varphi^* y)_p &= y_{\varphi(p)}. \end{aligned} \tag{3.4}$$

This linear map is a semiconjugacy between the corresponding admissible vector fields $\gamma_f^{\mathcal{N}_1}$ and $\gamma_f^{\mathcal{N}_2}$ with the same arbitrary response function $f \in \mathcal{C}^\infty(\bigoplus_{\sigma \in \Sigma_1} V, V)$:

$$\varphi^* \circ \gamma_f^{\mathcal{N}_2} = \gamma_f^{\mathcal{N}_1} \circ \varphi^*. \tag{3.5}$$

In particular, any integral curve $y(t)$ for the network $\mathcal{N}_2$ is sent to an integral curve $x(t) = \varphi^* y(t)$. Furthermore, the graph fibration induces the robust pattern of synchrony

$$\Delta_\varphi = \left\{ x \in \bigoplus_{p \in C_1} V \;\middle|\; x_p = x_q, \quad \textit{if} \quad \varphi(p) = \varphi(q) \right\}.$$

Any robust pattern of synchrony of $\mathcal{N}_1$ arises from a graph fibration into some other network in this way.

Remark 3.15. As we mentioned already in Section 2.4 the balanced partitions in Proposition 3.11 can be encoded in graph fibrations so that the existence of the corresponding synchrony spaces follows as a special case from Theorem 3.14. △

Once a graph fibration between networks is known, the dynamical implications can be investigated immediately via the induced semiconjugacy. It can also be useful not to seek for graph fibrations between two different networks. For example dependency between 'parts' of a network can be described in this language as well (compare to Example 2.7), which can be used to express feedforward structure entirely in terms of graph fibrations. Furthermore, one might extend a network to one with nice properties and draw conclusions on the original one via semiconjugacy. This is the underlying idea of the fundamental network that we introduce in the next parts which will also be the

[2] The authors state it as a reformulation of Theorems 3.8 and 3.11 in DEVILLE and LERMAN [25].

only instance of explicit usage of graph fibrations in this thesis. For more examples see Section 2.4 and consult DeVille and Lerman [24, 25, 26] and Nijholt, Rink, and Sanders [78]. It is not yet completely clear what other structural features can be encoded in graph fibrations the investigation of which is due to further research.

3.1.3 Semigroup networks

We make two additional assumptions on the set of input maps. As they are maps on the set of cells C, they can be concatenated $\tau \circ \sigma \colon C \to C$, which we abbreviate by $\tau\sigma$. This concatenation can be interpreted as another input map. A cell $p \in C$ receives an input of color σ from $\sigma(p)$. This in turn receives an input of color τ from $\tau(\sigma(p))$. Hence p receives what we call an *indirect input* of color $\tau\sigma$ from $(\tau\sigma)(p)$. The map $\tau\sigma$ describes the indirect input of this color of any cell. We use the same term – indirect inputs – for this construction following finitely many arrows backwards. However, in general $\tau\sigma$ need not be an element of Σ. Thus, we make the assumption that this is indeed the case, i.e. $\tau\sigma \in \Sigma$ for all $\sigma, \tau \in \Sigma$. This gives the set of input maps Σ the structure of a *semigroup* with multiplication or product given by concatenation of maps. Certainly, not every set of input maps has this property but it can be extended to a semigroup. There are multiple ways of doing so, all leading to the same result. The smallest semigroup containing Σ is the intersection of all semigroups containing Σ. It can be computed by including all possible products of finitely many elements in Σ. Then the resulting set is necessarily closed under taking products. The resulting set is also called the *closure* of Σ or the semigroup *generated* by Σ. If $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ we denote it by

$$\langle \Sigma \rangle = \langle \sigma_1, \dots, \sigma_n \rangle = \{\sigma_1, \dots, \sigma_n, \sigma_{n+1}, \dots \sigma_{n'}\}, \tag{3.6}$$

where $\sigma_{n+1}, \dots, \sigma_{n'}$ are all possible products of finitely many elements in Σ. In particular, for every $n + 1 \leq j \leq n'$ there exist $1 \leq i_1, \dots, i_{k_j} \leq n$ such that $\sigma_j = \sigma_{i_1} \cdots \sigma_{i_{k_j}}$. Note that the resulting semigroup is always finite as there are only finitely many transformations on the finite set C. We refer to the network with set of cells C and input maps $\langle \Sigma \rangle$ as the *(semigroup) extension* of the original network. The inclusion of all products of finitely many input maps means that we consider all the indirect inputs as direct inputs as well. This is in some sense a natural definition of networks. We discuss further reasons and implications on network dynamics shortly.

We usually assume that each cell depends on its own state. This can also be expressed in terms of input maps. To that end, we assume one of the input maps to be the identity on the set of cells $\mathrm{Id}_C \in \Sigma$. That means each cell receives an input of the same color from itself, as $\mathrm{Id}_C(p) = p$ for all $p \in C$ (we usually omit the corresponding arrow when depicting networks). Note that this map also acts as an identity in the semigroup, i.e. $\mathrm{Id}_C \circ \sigma = \sigma \circ \mathrm{Id}_C = \sigma$ for all $\sigma \in \Sigma$. Following this assumption, Σ has the structure of a monoid with identity $\mathrm{Id} = \mathrm{Id}_C$. Without loss of generality, we assume $\sigma_1 = \mathrm{Id}$, whenever we choose an ordering of the input maps. Hence, the first argument of the response function will always be the state variable of the cell itself.

In the remainder of this subsection we want to provide some additional intuition from Sections 3 and 7 in Rink and Sanders [92] why considering sets of input maps that are closed under concatenation is natural. To that end consider the network $\mathcal{N} = (C, \Sigma)$ and its extension $\mathcal{N}' = (C, \langle \Sigma \rangle)$. In particular, both networks have the same number of cells and, attaching the same internal phase space, also the same total phase space. This part only considers the extension to a semigroup. As the difference to a monoid of input maps depends only on a mere convention – Does the state of a cells depend on itself? –, which holds for the original network as well as its extension, we do not discuss these effects here. First of all, note that we can also extend admissible vector fields to the extended network. In particular, for a smooth response function $f \in \mathcal{C}^\infty(\bigoplus_{\sigma \in \Sigma} V, V)$ we define an extension $f' \in \mathcal{C}^\infty(\bigoplus_{\sigma \in \langle \Sigma \rangle} V, V)$ via

$$f'(y_1, \dots, y_n, y_{n+1}, \dots, y_{n'}) = f(y_1, \dots, y_n),$$

where n' is as in (3.6). Then obviously $\gamma_{f'}^{\mathcal{N}'} = \gamma_f^{\mathcal{N}}$, since

$$(\gamma_{f'}^{\mathcal{N}'})_p(x) = f'(x_{\sigma_1(p)}, \ldots, x_{\sigma_n(p)}, x_{\sigma_{n+1}(p)}, \ldots, x_{\sigma_{n'}(p)}) = f(x_{\sigma_1(p)}, \ldots, x_{\sigma_n(p)}) = (\gamma_f^{\mathcal{N}})_p(x)$$

for all $x \in \bigoplus_{p\in C} V$ and all $p \in C$. Hence, every network vector field of the original network can be seen as a vector field for the extended network. More precisely, the admissible vector fields of the original network form a subspace of the space of admissible vector fields for the extension.

Furthermore, the extension of the set of input maps has no effect on some of the most important structural features of a network, namely symmetry and synchrony. That is, a symmetry of the original network is also a symmetry of its extension and vice versa. Accordingly, a pattern of synchrony is robust for the original network if and only if it is robust for its extension.

Lemma 3.16 (Lemma 7.1 in [92]). *Let $\mathbf{p}\colon C \to C$ be a permutation. Then $\mathbf{p}$ is a symmetry of $\mathcal{N}$ if and only if it is a symmetry of $\mathcal{N}'$.*

Lemma 3.17 (Lemma 7.3 in [92]). *Let $P = \{P_1, \ldots, P_r\}$ be a partition of the set of cells C and Δ_P the corresponding synchrony subspace as in Proposition 3.11. Then Δ_P is a robust synchrony subspace for $\mathcal{N}$ if and only if it is a robust synchrony subspace for $\mathcal{N}'$.*

Motivated by these facts, from now on, we assume all networks to be homogeneous coupled cell systems whose input set has the structure of a monoid unless stated otherwise.

Remark 3.18. The input networks in a semigroup network can readily be encoded in terms of input maps. For a fixed cell p it contains all cells $q \in C$ such that there is a path from q to p. This is equivalent to q having an indirect input into p. As we consider all indirect inputs as direct inputs, this can be expressed as

$$C_p = \{p\} \cup \{\sigma(p) \mid \sigma \in \Sigma\}.$$

When Σ is a monoid we do not need to add p separately

$$C_p = \{\sigma(p) \mid \sigma \in \Sigma\}.$$

△

3.1.4 Fundamental networks

Given a semigroup network, one may construct a second network that is related to the original one via graph fibrations. Its interaction structure is determined by properties of the monoid of input maps, i.e. by the multiplicative behavior of its elements.

Definition 3.19 (Section 10 in [92]). *Given a semigroup network $\mathcal{N} = (C, \Sigma)$ we define its* fundamental network $\widetilde{\mathcal{N}} = (\Sigma, \Sigma)$ *– i.e. it has cells labeled by Σ – in which input maps act by multiplication from the left. That is $\tau \in \Sigma$ receives an arrow of color σ from $\sigma\tau$.*

We attach the same internal phase space V and the same internal dynamics to the cells of the fundamental network – the coupling phase spaces coincide for both networks. The fundamental network vector fields *are defined as in* (3.1) *and are of the form*

$$\Gamma_f(X) = \begin{pmatrix} f(X_{\sigma_1\sigma_1}, \ldots, X_{\sigma_n\sigma_1}) \\ f(X_{\sigma_1\sigma_2}, \ldots, X_{\sigma_n\sigma_2}) \\ \vdots \\ f(X_{\sigma_1\sigma_n}, \ldots, X_{\sigma_n\sigma_n}) \end{pmatrix}, \tag{3.7}$$

where the coordinates $X = (X_\sigma)_{\sigma\in\Sigma} \in \bigoplus_{\sigma\in\Sigma} V$ are chosen accordingly.

Remark 3.20. The construction of the fundamental network including arrows between monoid elements according to multiplications from the left is known from algebra where it is called the *(left) Caley graph*. More precisely the fundamental network can be defined as $\widetilde{\mathcal{N}} = (\widetilde{C}, \widetilde{\Sigma})$, where $\widetilde{C} = \Sigma$ and $\widetilde{\sigma}\colon \widetilde{C} \to \widetilde{C}, \tau \mapsto \sigma\tau$. One easily verifies

(i) $\widetilde{\sigma} \circ \widetilde{\sigma'} = \widetilde{\sigma\sigma'}$ for all $\sigma, \sigma' \in \Sigma$,

(ii) $\widetilde{\sigma}(\mathrm{Id}) = \sigma\,\mathrm{Id} = \sigma \neq \sigma' = \sigma'\,\mathrm{Id} = \widetilde{\sigma}'(\mathrm{Id})$ for $\sigma \neq \sigma'$ so that $\widetilde{\sigma} \neq \widetilde{\sigma}'$.

Then (i) implies that $\sigma \mapsto \widetilde{\sigma}$ is a surjective homomorphism of semigroups – $\widetilde{\Sigma}$ is defined as its image. In particular, if Σ is a monoid, so is $\widetilde{\Sigma}$ with identity $\widetilde{\mathrm{Id}}$. On the other hand, (ii) shows that $\widetilde{\sigma} = \widetilde{\sigma}'$ if and only if $\sigma = \sigma'$. As a result, the homomorphism is injective. Hence, it is also bijective, the monoids are isomorphic and we identify $\widetilde{\Sigma}$ with Σ. △

Remark 3.21 (Remark 10.9 and Proposition 10.10 in Rink and Sanders [92]). If we drop the assumption that Σ is a monoid, we cannot make this identification any more. In particular observation (ii) of Remark 3.20 depends crucially on the identity element Id in the monoid. The homomorphism $\sigma \mapsto \widetilde{\sigma}$ need not be injective for a semigroup, i.e. $\widetilde{\sigma} = \widetilde{\sigma}'$ even though $\sigma \neq \sigma'$ is possible. This happens if σ and σ' have the same left multiplicative behavior: $\sigma\tau = \sigma'\tau$ for all $\tau \in \Sigma$. As a result, the arrow colors σ and σ' exhibit the same interaction structure in the fundamental network. This situation is not according to the formalism of semigroup networks. Note, however, that this issue can be avoided, if we assume that the original network has no cells without any outgoing arrows. Then the homomorphism of semigroups is injective and we may identify $\widetilde{\Sigma}$ and Σ. Note that we may remove cells without any outgoing arrows from a network as they do not have any impact on the dynamics of any other cells. This requires slight adaptation of the formalism but does not impose much of a restriction. As we focus on the case where Σ is a monoid as in Remark 3.20, we do not include any further details. For more information see the mentioned references and the discussion thereafter. △

Remark 3.22. The fundamental network construction depends only on the input maps but not on the cells of a network. Hence, two networks with the same – or isomorphic – semigroups of input maps have the same fundamental networks. In particular, this holds true in the cases from before where the semigroup of input maps of a fundamental network can be identified with that of the original network. We immediately see that the fundamental network is also its own fundamental network which motivates calling it 'fundamental'. △

Corollary 3.23. *Reformulating the definitions of adjacency matrices in Definition 3.6 to fundamental networks is interesting in its own right. For these the adjacency matrices B_σ are defined via multiplication from the left in Σ, i.e. $(B_\sigma(X))_\tau = X_{\sigma\tau}$. Additionally, we may compute that the B_σ respect the multiplicative structure of Σ, i.e. $B_{\mathrm{Id}} = \mathbb{1}_{\bigoplus_{\sigma\in\Sigma} V}$ – the identity on $\bigoplus_{\sigma\in\Sigma} V$ – and $B_\sigma B_\tau = B_{\tau\sigma}$. Once again these adjacency matrices span the subspaces of all linear network vector fields as in Proposition 3.8 and Corollary 3.9.*

Not only through its construction, but also in terms of dynamics, the fundamental network is closely related to the original network. This can be deduced from the following theorem, which introduces graph fibrations from the fundamental network to the original network and explores dynamical consequences according to Theorem 3.14.

Theorem 3.24 (Theorem 6.4 in [78])**.** *Let $\mathcal{N} = (C, \Sigma)$ be a network and $\widetilde{\mathcal{N}} = (\Sigma, \Sigma)$ its fundamental network. For every cell $p \in C$ the map $\varphi_p\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ defined by*

$$\varphi_p(\sigma) = \sigma(p) \tag{3.8}$$

where $\sigma \in \Sigma$ is a cell of $\widetilde{\mathcal{N}}$, is a graph fibration. The induced linear semiconjugacy $\varphi_p^* \circ \gamma_f = \Gamma_f \circ \varphi_p^*$ mapping the total phase space of $\mathcal{N}$ to that of $\widetilde{\mathcal{N}}$ is given by

$$\begin{aligned} \varphi_p^* \colon \bigoplus_{p \in C} V &\to \bigoplus_{\sigma \in \Sigma} V \\ (x_1, \ldots, x_N) &\mapsto (x_{\sigma_1(p)}, \ldots, x_{\sigma_n(p)}) \end{aligned} \tag{3.9}$$

We briefly summarize some of the main implications for the dynamics of the two networks and their relations. First of all note that a semiconjugacy maps integral curves – i.e. solutions to the corresponding homogeneous coupled cell systems – to integral curves. In particular, if $x(t)$ solves the ordinary differential equation of the original network $\dot{x} = \gamma_f(x)$, then $X(t) = \varphi_p^*(x(t))$ for any $p \in C$ solves the equation for the fundamental network, $\dot{X} = \Gamma_f(X)$. On the other hand the image of the graph fibration φ_p is precisely the input network $\mathcal{N}_p$ of p, i.e.

$$\operatorname{im}(\varphi_p) = \{\sigma(p) \mid \sigma \in \Sigma\} = C_p.$$

As a matter of fact, we could have used the induced maps $\varphi_p^* \colon \bigoplus_{p \in C} V \to \bigoplus_{\sigma \in \Sigma} V$ in order to define admissible vector fields for the original networks

$$(\gamma_f)_p(x) = f(x_{\sigma_1(p)}, \ldots, x_{\sigma_n(p)}) = f(\varphi_p^*(x)) = (f \circ \varphi_p^*)(x).$$

Since input networks are proper subnetworks of $\mathcal{N}$ (compare also to Remark 4.15 in Aguiar, Dias, and Soares [10]), we can also investigate the dynamics of $\mathcal{N}$ restricted to $\mathcal{N}_p$. Theorem 3.24 together with Theorem 3.14 amount to

Corollary 3.25 (Corollary 6.6 in [78]). *The dynamics of the input network $\mathcal{N}_p$ of any cell $p \in C$ is embedded in the dynamics of the fundamental network $\mathcal{N}$ as the robust synchrony subspace*

$$\Delta_p = \Delta_{\varphi_p} = \left\{ X \in \bigoplus_{\sigma \in \Sigma} V \;\middle|\; X_\sigma = X_\tau, \quad \textit{if} \quad \sigma(p) = \tau(p) \right\}.$$

Since the evolution of x_p depends only on the state variables of cells in the input network $\mathcal{N}_p$, this allows to recover the dynamics of any cell in the original network from solutions in the robust synchrony subspace from Corollary 3.25. More precisely, given initial conditions $x(0) \in \bigoplus_{p \in C} V$ for $\mathcal{N}$ and solutions $X_{(p)}(t)$ with $X_{(p)}(0) = \varphi_p^*(x(0))$ – i.e. initial conditions in the synchrony subspace Δ_p corresponding to the initial conditions in $\mathcal{N}_p$ – for $\widetilde{\mathcal{N}}$, we may compute the solution $x(t) = (x_p(t))_{p \in C}$ by integration of

$$\dot{x}_p(t) = f(X_{(p)}(t))$$

for all $p \in C$ (compare also to Theorem 10.1 and thereafter in Rink and Sanders [92]).

A direct consequence of Corollary 3.25, is that the dynamics of the entire original network is embedded as a robust synchrony pattern in those of the fundamental network, if there is a cell whose input network is the entire network, i.e. $C_p = C$ for some $p \in C$. More precisely speaking, this is the case, when there is a cell $p \in C$ such that for every $q \in C$ there is an input map $\sigma \in \Sigma$ with $\sigma(p) = q$, meaning p receives an input from any other cell in the network. In that case, it is not hard to see, that the partition of the cells of the fundamental network according to the synchrony pattern Δ_p is balanced. Hence, the original network is a quotient of the fundamental network in the sense of the groupoid formalism. On the other hand, it can be shown that this is also the only case in which the original network is a quotient of its fundamental network (see Proposition 5.7 in Aguiar, Dias, and Soares [10]). We also recommend that reference for more details on structural properties of the fundamental network in comparison with the original network. Note that Nijholt, Rink, and Sanders [78] generalize the notion of quotient networks by calling $\mathcal{N}_1$ a quotient of $\mathcal{N}_2$ if there is a graph fibration $\varphi \colon \mathcal{N}_2 \to \mathcal{N}_1$. In that sense $\mathcal{N}$ is always a quotient of its fundamental network $\widetilde{\mathcal{N}}$.

3.1.5 Hidden symmetry

The dynamical relations to the original network are only part of the reason, why fundamental networks are an important concept. It turns out, that fundamental networks are subject to generalized symmetries. That is, they are not symmetric in the classical sense, but allow for so-called *self-fibrations*, i.e. graph fibrations from the fundamental network to itself – which have similar implications for their dynamics. Once again, let $\widetilde{\mathcal{N}}$ denote a fundamental network of a network whose input maps form a monoid. Recall that $\widetilde{\mathcal{N}}$ is also its own fundamental network. Hence, we may apply Theorem 3.24 to $\widetilde{\mathcal{N}}$ and obtain

Theorem 3.26 (Theorem 7.2 in [78]). *Let $\widetilde{\mathcal{N}} = (\Sigma, \Sigma)$ be a fundamental homogeneous coupled cell network, where Σ is a monoid. For every $\tau \in \Sigma$ the map $\varphi_\tau \colon \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$ defined by*

$$\varphi_\tau(\sigma) = \sigma\tau \tag{3.10}$$

where $\sigma \in \Sigma$ is a cell of $\widetilde{\mathcal{N}}$, is a graph fibration. The induced linear semiconjugacy

$$\varphi_\tau^* \circ \Gamma_f = \Gamma_f \circ \varphi_\tau^*. \tag{3.11}$$

is given by

$$\begin{aligned} \varphi_\tau^* \colon \bigoplus_{\sigma\in\Sigma} V &\to \bigoplus_{\sigma\in\Sigma} V \\ \varphi_\tau^*(X) &\mapsto (X_{\sigma\tau})_{\sigma\in\Sigma}. \end{aligned} \tag{3.12}$$

Remark 3.27. Due to their importance in the remainder of this text, we denote the induced linear maps by $A_\sigma = \varphi_\sigma^*$ for all $\sigma \in \Sigma$. Similar to the adjacency matrices they can be interpreted as $n \times n$ matrices with entries in $\mathfrak{gl}(V)$ that are defined by the right multiplicative behavior of in the monoid $(A_\sigma(X))_\tau = X_{\tau\sigma}$. These could have been used to define the fundamental network vector fields as before

$$(\Gamma_f)_\sigma(X) = f(X_{\sigma_1\sigma}, \dots, X_{\sigma_n\sigma}) = f(A_\sigma X) = (f \circ A_\sigma)(X).$$

△

The fundamental network is subject to non-trivial self-fibrations. Contrary to classical symmetries, these do not need to be invertible. The corresponding semiconjugacies of the fundamental network vector fields with themselves, however, amount to the equivariance condition (3.9) in just the same way as classical symmetries to (compare to (3.3)). Interestingly, the fundamental network structure is entirely encoded in these symmetries.

Theorem 3.28 (Theorem 3.11 in [91]). *Let $\widetilde{\mathcal{N}} = (\Sigma, \Sigma)$ be a fundamental homogeneous coupled cell network, where Σ is a monoid.*

(i) The symmetries satisfy

$$\begin{aligned} A_{\mathrm{Id}} &= \mathbb{1}_{\bigoplus_{\sigma\in\Sigma} V} \\ A_\sigma A_\tau &= A_{\sigma\tau}, \end{aligned}$$

for all $\sigma, \tau \in \Sigma$, where $\mathbb{1}_{\bigoplus_{\sigma\in\Sigma} V}$ is the identity on $\bigoplus_{\sigma\in\Sigma} V$.

(ii) Each fundamental network Γ_f is equivariant with respect to these symmetries

$$\Gamma_f \circ A_\sigma = A_\sigma \circ \Gamma_f$$

for all $f \in \mathcal{C}^\infty(\bigoplus_{\sigma\in\Sigma} V, V)$ and $\sigma \in \Sigma$.

(iii) On the other hand, if $F \in \mathcal{C}^\infty(\bigoplus_{\sigma\in\Sigma} V, \bigoplus_{\sigma\in\Sigma} V)$ satisfies

$$F \circ A_\sigma = A_\sigma \circ F$$

for all $\sigma \in \Sigma$, then F is a fundamental network vector field, i.e. there exists $f \in \mathcal{C}^\infty(\bigoplus_{\sigma\in\Sigma} V, V)$ such that $F = \Gamma_f$.

The multiplicative behavior of the symmetries A_σ described in (i) yield that $\sigma \mapsto A_\sigma \in \mathfrak{gl}(\bigoplus_{\sigma\in\Sigma} V)$ is a homomorphism of monoids. Hence, $(\bigoplus_{\sigma\in\Sigma} V, \sigma \mapsto A_\sigma)$ is a *representation* of the monoid Σ. When Σ is a group this matches the well-known definition of a group representation. If the context is clear, we also refer to the underlying representation space as the representation. Points (ii) and (iii) yield that the fundamental homogeneous coupled cell systems can entirely be expressed in terms of these symmetries. More precisely, once one has settled on an internal phase space V, the admissible vector fields on the total phase space are precisely the ones that are equivariant under the monoid representation given by $\sigma \mapsto A_\sigma$

$$\left\{\Gamma_f \;\middle|\; f \in \mathcal{C}^\infty\left(\bigoplus_{\sigma\in\Sigma} V, V\right)\right\} = \left\{F \in \mathcal{C}^\infty\left(\bigoplus_{\sigma\in\Sigma} V, \bigoplus_{\sigma\in\Sigma} V\right) \;\middle|\; F \circ A_\sigma = A_\sigma \circ F \text{ for all } \sigma \in \Sigma\right\}. \tag{3.13}$$

Remark 3.29. The representation matrices are defined by multiplication from the right in the monoid $(A_\sigma(X))_\tau = X_{\tau\sigma}$. Sometimes we use the term *(right) regular representation of Σ on $\bigoplus_{\sigma\in\Sigma} V$*. Recall from Corollary 3.23 that the adjacency matrices of the fundamental network are defined by multiplication from the left $(B_\sigma(X))_\tau = X_{\sigma\tau}$ and satisfy $B_{\mathrm{Id}} = \mathbb{1}_{\bigoplus_{\sigma\in\Sigma} V}$ as well as $B_\sigma B_\tau = B_{\tau\sigma}$. Consequently, these matrices define what we call the *(left) regular (anti-)representation*. The term *anti* highlights the reversed order of the multiplication.

In Corollary 3.23 we have seen that the subspace of linear admissible fundamental network vector fields are spanned by these adjacency matrices. Moreover, due to the equivalence of admissibility and equivariance (3.13) these are exactly the linear maps that commute with the representation matrices. These are of enormous importance in the representation theory of monoids, where they are called *endomorphisms of the representation* $\mathrm{End}_\Sigma\left(\bigoplus_{\sigma\in\Sigma} V\right)$. In particular, in the case $V = \mathbb{R}$ we have

$$\mathrm{End}_\Sigma\left(\bigoplus_{\sigma\in\Sigma} \mathbb{R}\right) = \langle B_\sigma \mid \sigma \in \Sigma\rangle$$

which denotes the real algebra spanned by the adjacency matrices B_σ due to Corollary 3.9. We will explore the necessary background of monoid representations in Chapter 4. △

In the very beginning of this thesis we described one of the main problems in network dynamics as the fact, that network structure is not stable under coordinate transformations. Since qualitative dynamical features are those that are robust under such transformations, it is difficult to determine dynamical properties that are inherent to a given network. Symmetry, on the other hand, is not lost under coordinate transformations, the representation matrices transform accordingly and the equivariance condition remains intact. The equivalence of network structure and (generalized) symmetry of the fundamental network enables us to forego this issue. To that end, we may characterize dynamical features by investigating the restrictions imposed by the symmetries on equivariant systems. These are inherent to the fundamental network. This even allows for a classification of network dynamics in terms of their symmetries.

Note that these generalized symmetries do not necessarily act non-trivially on the robust synchrony subspaces that correspond to (the input networks of) the original network $\mathcal{N}$. Hence, their effect might not be seen directly in the ordinary differential equations of the original network. Nevertheless, they impact the dynamics of the fundamental network altogether which, in turn, restricts the dynamics of the original network due to the graph fibrations. Hence, we refer to the symmetries of the fundamental network as *hidden symmetries* of the original network.

In classical equivariant dynamics one usually considers representations of (finite or compact Lie) groups and investigates phenomena that are inherent to the imposed restrictions. As monoids have less structure than groups so do their representations. Nevertheless, many tools from equivariant dynamics can be adapted to this more general setting. We highlight some techniques that are especially useful in bifurcation analysis in the upcoming sections.

3.2 Bifurcation problems

The main results of this thesis are involved with the study of bifurcations in network dynamical systems. Oftentimes, one is not only interested in the dynamical features of a given system but more so in transitions between them. That is, given a specific solution, what kind of solutions will a slightly varied system exhibit. In most cases, the dynamics are essentially the same in both cases. We are, however, interested in those situations where qualitative changes – i.e. a *bifurcation* – occur. Usually, we assume the existence of a fully synchronous steady state solution that changes its stability properties and investigate variations in the dynamical behavior – e.g. number of steady states, emergence of periodic solutions – locally. If newly appearing solutions exhibit less synchrony, we speak of a *synchrony breaking bifurcation*. This is according to similar investigations in equivariant dynamics, where bifurcating solutions are not fully *symmetric* so that the system undergoes a *symmetry breaking bifurcation*.

Let us fix some notation for this situation. In order to drive variations of the system, we assume that the internal dynamics additionally depends on a parameter $\lambda \in \Lambda \subset \mathbb{R}^l$, where Λ is an open subset. As we focus on a local investigation, the extension to $\Lambda = \mathbb{R}^l$ usually imposes no restriction of generality. If $l \geq 2$ we also speak of multiple parameters and interpret every coordinate entry of λ as a parameter in its own right. Then we extend the response function to $f \in \mathcal{C}^\infty(\bigoplus_{\sigma\in\Sigma} V \times \Lambda, V)$ so that the admissible vector fields are

$$\gamma_f(x,\lambda) = \begin{pmatrix} f(x_{\sigma_1(p_1)}, \dots, x_{\sigma_n(p_1)}, \lambda) \\ f(x_{\sigma_1(p_2)}, \dots, x_{\sigma_n(p_2)}, \lambda) \\ \vdots \\ f(x_{\sigma_1(p_N)}, \dots, x_{\sigma_n(p_N)}, \lambda) \end{pmatrix}. \tag{3.14}$$

We may express the response functions as a family $f_\lambda = f(\cdot, \lambda)$. The collection of vector fields $\gamma_{f_\lambda,\lambda} = \gamma_f(\cdot,\lambda)$ for all $\lambda \in \Lambda$ is called an *l-parameter family of admissible vector fields*. The system of ordinary differential equations governed by a parameter dependent vector field is a *parameter dependent network coupled cell system*. Note that the constructions for fundamental networks are completely analogous. Hence, we use the notation $\Gamma_{f_\lambda,\lambda} = \Gamma_f(\cdot,\lambda)$ accordingly. The results on network vector fields and their induced dynamics that we state the previous section hold for parameter dependent vector fields in the sense that they are true for $\gamma_{f_\lambda,\lambda}$ or $\Gamma_{f_\lambda,\lambda}$ for any fixed $\lambda \in \Lambda$.

We assume the existence of a fully synchronous equilibrium solution $x_0 = (v, \dots, v) \in \bigoplus_{p\in C} V$ at a parameter value $\lambda_0 \in \Lambda$:

$$\gamma_f(x_0, \lambda_0) = 0. \tag{3.15}$$

By translation of coordinates, we may assume the point of interest to be the origin without loss of generality, i.e. $x_0 = 0$ and $\lambda_0 = 0$. Without explicitly mentioning it, we focus on a small enough neighborhood around x_0 that does not contain any other steady state or periodic solutions for $\lambda = \lambda_0$. The (linear) stability properties of x_0 are determined by spectral properties of the linearization of the vector field. In particular, linearization with respect to the spatial directions $D_x\gamma_f(x_0,\lambda_0)$. If it has no eigenvalues on the imaginary axis, $\operatorname{spec}(D_x\gamma_f(x_0,\lambda_0)) \cap \mathbf{i}\mathbb{R} = \emptyset$, the steady state is said to be *hyperbolic*. It is well known, that a system is *structurally stable* near a hyperbolic steady state. In particular, qualitatively the solutions remain intact under small variations of the parameter λ. Hence, in order for a bifurcation – a change in qualitative dynamics – to occur, the existence of an eigenvalue on the imaginary axis is a necessary condition. In that case, (x_0, λ_0) is said to be a *bifurcation point*. The eigenvalues on $\mathrm{i}\mathbb{R}$ are also called *critical* or *central*.

We distinguish two types of bifurcations. *Steady state bifurcations* describe changes only in the number and properties of steady state solutions to the coupled cell system. That is, we wish to determine solutions to

$$\gamma_f(x, \lambda) = 0 \tag{3.16}$$

for (x, λ) close to (x_0, λ_0) and to describe their properties. If $D_x\gamma_f(x_0, \lambda)$ has only non-vanishing eigenvalues, the implicit function theorem implies that (3.16) has a unique *branch* of fully synchronous solutions $(x(\lambda), \lambda)$. Hence, in order to observe non-trivial branches we do not only need eigenvalues on the imaginary axis, but at least one eigenvalue 0. Most of our results aim at the investigation of steady state bifurcations.

Secondly, we are interested in determining periodic solutions to

$$\dot{x} = \gamma_f(x, \lambda) \tag{3.17}$$

for (x, λ) close to (x_0, λ_0). This type of qualitative change in dynamical behavior is also known as *Hopf bifurcation*. Due to the Hopf bifurcation theorem, a periodic solution can only occur via a bifurcation if the linearization $D_x\gamma_f(x_0, \lambda_0)$ has a pair of purely imaginary eigenvalues. In both cases we refer to solutions that emerge through a bifurcation as *branching* solutions. Furthermore, we may parameterize a curve describing the branching steady states or the initial conditions of branching periodic solutions smoothly by the parameter λ. We refer to these curves as *(bifurcating) branches*.

We are interested in bifurcations that are dictated by the network structure. Hence, we do not fix a response function f but rather pursue the investigation for all possible internal dynamics. In order to exclude special cases that only occur for specific choices of internal dynamics, we focus on *generic* response functions. Genericity usually denotes an open and dense subset of $\mathcal{C}^\infty(\bigoplus_{\sigma\in\Sigma} V \times \Lambda, V)$ in the $\mathcal{C}^\infty$-topology and can be thought of as a mathematical notion of a 'normal case'. This will become more obvious in later parts of the text, where we characterize genericity in terms of certain expressions of *system parameters* – i.e. partial derivatives or Taylor coefficients of f –, which determines a dense open subset of the set of system parameters. The underlying idea in all possible realizations of this concept is that we want to describe bifurcations that are inherently dictated by the network structure but are not only possible because of one specific choice of governing functions.

Certainly, both situations described above can occur simultaneously: $D_x\gamma_f(x_0, \lambda_0)$ may for example have one eigenvalue 0 and a pair of purely imaginary eigenvalues. In that case, we speak of *mode interactions*, as both bifurcations, in some sense, occur at the same time and therefore interact. In 1-parameter bifurcations of systems without any additional structure mode interactions do not occur generically. Under the given assumptions the linearization either has a simple eigenvalue 0 or a pair of simple eigenvalues on the imaginary axis. Thus, there is either only a steady state bifurcation or a Hopf bifurcation. However, additional structure, such as symmetry or an underlying network, is known to imply spectral degeneracies, e.g. eigenvalues with high multiplicities, so that phenomena differ substantially.

Remark 3.30. Note that for an l-parameter family of vector fields an open and dense subset of the bifurcation points (in the subset topology of $\mathbb{R}^l$) essentially only depends on one parameter. That is, a small perturbation of all but one parameter yields a small variation of the remaining one such that the critical eigenvalue structure remains the same – 0 has the same multiplicity, pairs of purely imaginary eigenvalues vary along the imaginary axis but keep their multiplicities as well – due to the implicit function theorem. This leads to hyperplanes of bifurcation points in which the linearization has critical eigenvalues that are due to only one parameter. It is only in their intersections, that critical eigenvalues depend on multiple parameters being in a specific setting and only there mode interactions can occur. Consequently, we only refer to an intersection of all these hyperplanes – in a point where the configuration of critical eigenvalues depends on all l parameters –

as the l-parameter bifurcation point. Thus, in order to describe the full (generic) bifurcation behavior of an l-parameter family of vector fields one has to investigate k-parameter bifurcations for all $k = 1, \dots, l$. In particular, the 1-parameter bifurcations can be seen as the elementary constituents of the full bifurcation picture which makes them specifically important (in GOLUBITSKY and STEWART [53] they are referred to as "The most important local bifurcations [...]"). As an illustrating example consider a 2-parameter family of vector fields $F\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ with a steady state $x(\lambda)$ for all $\lambda \in \mathbb{R}^2$ and assume that the linearization is of the form

$$\begin{pmatrix} \lambda_1 & \bullet \\ 0 & \lambda_2 \end{pmatrix}.$$

Then this family has eigenvalues λ_1 and λ_2. In particular, we obtain two curves of bifurcation points given by $(0, \lambda_2)$ and $(\lambda_1, 0)$ for $\lambda_1, \lambda_2 \in \mathbb{R}$. Only in their intersection point $(\lambda_1, \lambda_2) = (0,0)$ does the linearization exhibit a critical eigenvalue 0 of multiplicity 2. Whenever only one of the parameters vanishes this multiplicity is 1 independent of the other parameter. Hence, only in $(\lambda_1, \lambda_2) = (0,0)$ a 2-parameter bifurcation occurs, whereas all other points on these curves amount to 1-parameter bifurcations. $\triangle$

A central part of the investigations in the remainder of this thesis is devoted to the investigation of spectral properties of linearizations. As these are again admissible vector fields for the network in consideration, this highlights the importance of the results in Proposition 3.8 and Corollary 3.23, which provide knowledge about how the interaction structure translates into linear admissible network vector fields. In the remainder of this chapter we briefly summarize some of the classical reduction methods for bifurcation analysis and their reformulations for semigroup networks.

3.3 Normal forms

A powerful tool for bifurcation analysis as described in Section 3.2 is the so-called *normal form reduction*. The main idea is to find a coordinate transformation that allows to represent the (parameter dependent) vector fields in 'simpler' expressions while still capturing all the qualitative dynamical features such as steady state or periodic solutions. This, however, leaves room for interpretation in particular as to what we mean by 'simpler' expressions. However, one way of computing normal forms is to consecutively set up a transformation via Lie brackets of admissible vector fields that guarantees for the homogeneous polynomials in a Taylor expansion to only come from specific subspaces and truncate the procedure at some degree. We only briefly summarize the main results here, since we do not explicitly use normal forms in this thesis.

Note that the semigroup structure plays a crucial role. In general, admissible vector fields do not form a Lie algebra so that one cannot expect the normal form computed in this way to be admissible again. Hence, it is not suitable for the investigation of dynamical behavior that is inherent to the network structure. On the other hand, admissible vector fields of semigroup networks allow for a Lie algebra structure which is defined in terms of the response functions. First, we can see that the concatenation of two admissible vector fields γ_f and γ_g yields an admissible vector field as in

$$\gamma_f \circ \gamma_g = \gamma_{f \circ_\Sigma g},$$

where $f \circ_\Sigma g\colon \bigoplus_{\sigma\in\Sigma} V \to V$ is defined as

$$f \circ_\Sigma g = f \circ \Gamma_g = f \circ ((g \circ A_{\sigma_1}) \times \cdots \times (g \circ A_{\sigma_n}))$$

using Remark 3.27 (see Theorem 4.2 in RINK and SANDERS [92]). This shows that admissible vector fields for semigroup networks are closed under concatenations. Accordingly, we define a Lie bracket on response functions as

$$[f,g]_\Sigma(x) = (df \cdot \Gamma_g - dg \cdot \Gamma_f)(x) = (df(x) \circ_\Sigma g)(x) - (dg(x) \circ_\Sigma f)(x),$$

where $df(x), dg(x)$ denote the total derivatives interpreted as linear maps $\bigoplus_{\sigma\in\Sigma} V \to V$. Then the classical Lie bracket on admissible vector fields satisfies

$$[\gamma_f, \gamma_g](x) = D\gamma_f(x) \cdot \gamma_g(x) - D\gamma_g(x) \cdot \gamma_f(x) = \gamma_{[f,g]_\Sigma}(x).$$

Hence, the homogeneous coupled cell network vector fields are closed under this operation which provides the structure of a Lie algebra (see Theorem 5.1 and Lemma 5.2 in RINK and SANDERS [92]). Note that for parameter dependent vector fields the definitions need to be slightly adapted to the families of vector fields, i.e. for $f, g \in \mathcal{C}^\infty(\bigoplus_{\sigma\in\Sigma} V \times \Lambda, V)$ define

$$f \circ_\Sigma g = f_\lambda \circ \Gamma_{g_\lambda} \quad \text{and} \quad [f,g]_\Sigma(x) = (df_\lambda(x) \circ_\Sigma g_\lambda)(x) - (dg_\lambda(x) \circ_\Sigma f_\lambda)(x)$$

for the families $f_\lambda, g_\lambda \colon \bigoplus_{\sigma\in\Sigma} V \to \bigoplus_{\sigma\in\Sigma} V$.

Theorem 3.31 (Theorem 6.2 in [92]). *Let $f \in \mathcal{C}^\infty(\bigoplus_{\sigma\in\Sigma} V \times \Lambda, V)$ be a smooth response function with $f(0,0) = 0$ and Taylor expansion*

$$f = (f_{-1,1} + f_{-1,2} + \cdots) + (f_{0,0} + f_{0,1} + \cdots) + (f_{1,0} + f_{1,1} + \cdots) + \cdots,$$

where

$$f_{i,j} \in \mathcal{P}^{i,j} = \left\{ P \colon \bigoplus_{\sigma\in\Sigma} V \to V \;\middle|\; \text{homogeneous polynomial of degree } i+1 \text{ in } X \text{ and } j \text{ in } \lambda \right\}.$$

Fix $r, s \in \mathbb{N}$ and for all $-1 \le i \le r$ and $0 \le j \le s$ let $N^{i,j} \subset \mathcal{P}^{i,j}$ be a subspace such that

$$N^{i,j} \oplus \operatorname{im}\left(\operatorname{ad}^\Sigma_{f_{0,0}}\Big|_{\mathcal{P}^{i,j}}\right) = \mathcal{P}^{i,j},$$

where

$$\operatorname{ad}^\Sigma_{f_{0,0}} = [f_{0,0}, \cdot]_\Sigma \colon \quad \mathcal{C}^\infty\left(\bigoplus_{\sigma\in\Sigma} V \times \Lambda, V\right) \to \mathcal{C}^\infty\left(\bigoplus_{\sigma\in\Sigma} V \times \Lambda, V\right)$$

– in particular $\operatorname{ad}^\Sigma_{f_{0,0}}(\mathcal{P}^{i,j}) \subset \mathcal{P}^{i,j}$.

Then there exists a polynomial family Φ_λ of analytic diffeomorphisms for λ close to 0, conjugating $\gamma_{f_\lambda,\lambda}$ to $\gamma_{\overline{f}_\lambda,\lambda}$ on an open neighborhood of 0. Therein

$$\overline{f} = (\overline{f}_{-1,1} + \overline{f}_{-1,2} + \cdots) + (f_{0,0} + \overline{f}_{0,1} + \cdots) + (\overline{f}_{1,0} + \overline{f}_{1,1} + \cdots) + \cdots,$$

where

$$\overline{f}_{i,j} \in N^{i,j}$$

for all $-1 \le i \le r$ and $0 \le j \le s$.

The parameter dependent vector field $\gamma_{\overline{f}}$ is the *normal form* of γ_f around $(0,0)$ (compare to the discussion of bifurcation points in Section 3.2). As it is conjugate to the original vector field – for every parameter value close to the bifurcation point – both exhibit the same qualitative dynamics. Note that the normal form is defined as an admissible vector field with response function $\overline{f}$. In particular, this implies that the normal form of a fundamental network exhibits the same (monoid) symmetries as the full system. The function $\overline{f}$ can be seen as a normal form for the internal dynamics f. Its Taylor coefficients satisfy $\overline{f}_{i,j} \in N^{i,j} \subset \mathcal{P}^{i,j}$. These subspaces determine how much 'simpler' this normal form can actually be. As a matter of fact, identifying the subspaces $N^{i,j}$ as complements of the function given by the Lie bracket of the linearization $f_{0,0} = df(0,0)$ requires solving the so-called *homological equations* which are the main technical difficulty in the computation of normal forms. Furthermore, it might not be all that easy to extract the information on bifurcations one is interested in from the normal form – 'simpler' does not necessarily mean simple. Hence, we do not go into any more detail here but refer to the proof of the stated theorem and the examples provided in the same reference to obtain more information. Note that normal form computations like these have been applied to classify Hopf bifurcations in feedforward chains in RINK and SANDERS [90].

3.4 Lyapunov-Schmidt reduction

Another, easier to apply, method to analyze bifurcations is the *Lyapunov-Schmidt reduction*. This technique is well-known for general dynamical systems, where it has proven to be particularly useful in steady state bifurcation analysis.[3] It allows to reduce the investigation of bifurcating steady state solutions (compare to (3.16)) on the full state space to an equation only on the kernel of the linearization at the bifurcation point – recall that an eigenvalue 0 is a necessary condition for a steady state bifurcation. In general, this greatly reduces the dimension of the problem which simplifies computations. All branching steady state solutions can be found in the reduced equations as well as the direction in which they emerge from the fully synchronous solution – when denoted as a function of the parameter λ. Their stability properties, on the other hand, can not be determined. Since we want to investigate bifurcations that are caused by additional structure in the equations, we need to translate this structure onto the kernel and into the reduced equation. This, however, is too much to ask in the case of networks. As the kernel of a linear map – even an admissible one – usually comes with its own coordinates, it is simply not obvious how the network structure can be expressed in terms of the kernel. On the other hand, it is well known from classical equivariant dynamics that (group) symmetries can be carried over to the kernel of a linearized equivariant vector field. The representation matrices leave kernel and its complement, the image, invariant. This allows to construct the reduced system such that it is equivariant as well.

A similar result can be obtained for monoid equivariant dynamics as well. The technique needs to be adapted only slightly. Due to the equivariance condition (3.13) it can then be applied directly to bifurcation problems in fundamental network vector fields. Since the monoid symmetries are equivalent to the fundamental network structure, this allows to translate the network's impact to the kernel and determine bifurcating solutions that are forced by the network. To that end let $\mathbf{V}$ be a finite-dimensional real vector space and Θ a monoid that is represented on $\mathbf{V}$, i.e. there is a homomorphism of monoids $\theta \mapsto T_\theta \in \mathfrak{gl}(\mathbf{V})$. Furthermore, we reformulate the bifurcation problem by investigating a parameter dependent equivariant vector field $F \in \mathcal{C}^\infty(\mathbf{V} \times \Lambda, \mathbf{V})$ with $F(T_\theta X, \lambda) = T_\theta F(X, \lambda)$ for all $X \in \mathbf{V}, \lambda \in \Lambda, \theta \in \Theta$. We seek to determine all solutions to

$$F(X, \lambda) = 0$$

close to (X_0, λ_0), where $F(X_0, \lambda_0) = 0$ and $L = D_X F(X_0, \lambda_0)$ has an eigenvalue 0. Furthermore, we assume that X_0 is fully symmetric, meaning $T_\theta X_0 = X_0$ for all $\theta \in \Theta$. One readily sees that L is equivariant as well, i.e. $L \circ T_\theta = T_\theta \circ L$ for all $\theta \in \Theta$. For generalized symmetries by a monoid, the representation does not leave the kernel and image of L invariant. They do, however, leave generalized eigenspaces invariant. Hence, when we define

$$\ker_0(L) = \mathbb{E}_0, \quad \operatorname{im}_0(L) = \bigoplus_{\mu \in \operatorname{spec}(L) \setminus \{0\}} \mathbb{E}_\mu,$$

where $\mathbb{E}_\mu$ is the generalized eigenspace to the eigenvalue μ of L. We obtain

$$\ker_0(L) \oplus \operatorname{im}_0(L) = \mathbf{V}$$

but more importantly

$$T_\theta \ker_0(L) \subset \ker_0(L), \quad T_\theta \operatorname{im}_0(L) \subset \operatorname{im}_0(L)$$

for all $\theta \in \Theta$. We refer to these subspaces as the *generalized kernel* and the *reduced image* of L and denote the corresponding projections along $\operatorname{im}_0(L)$ and along $\ker_0(L)$ respectively by

$$P^c \colon \mathbf{V} \to \ker_0(L), \quad P^h \colon \mathbf{V} \to \operatorname{im}_0(L).$$

[3] It is also central for the proof of the Hopf bifurcation theorem. Its application, however, is not as straightforward as for steady state bifurcations.

As the subspaces are invariant under the representation, these projections are equivariant. Any element $X \in \mathbf{V}$ can uniquely be split as $X = X^c + X^h$ where $X^c = P^c(X) \in \ker_0(L)$ and $X^h = P^h(X) \in \operatorname{im}_0(L)$. Likewise, we see that $F(X, \lambda) = 0$ if and only if

$$F^c(X^c + X^h, \lambda) = P^c(F(X^c + X^h, \lambda)) = 0 \quad \text{and} \tag{3.18a}$$
$$F^h(X^c + X^h, \lambda) = P^h(F(X^c + X^h, \lambda)) = 0 \tag{3.18b}$$

and we solve them consecutively. As a matter of fact, $F^{c,h}(X_0^c + X_0^h, \lambda_0) = 0$. We begin with (3.18b) and note that $F^h\colon \ker_0(L) \oplus \operatorname{im}_0(L) \to \operatorname{im}_0(L)$. The derivative in the direction of $\operatorname{im}_0(L)$ is given by

$$D_{X^h} F^h(X_0, \lambda_0) = P^h \circ D_{X^h} F(X_0, \lambda_0) = P^h \circ \left(L|_{\operatorname{im}_0(L)} \right) : \operatorname{im}_0(L) \to \operatorname{im}_0(L),$$

which is clearly invertible, as L is restricted to the direct sum of all its generalized eigenspaces with non-vanishing eigenvalues and P^h is the projection onto that subspace. Hence, we may apply the implicit function theorem to obtain a smooth function $X^h = X^h(X^c, \lambda)$ for X^c close to X_0^c and λ close to λ_0 that uniquely determines all solutions of (3.18b) close to X_0^h:

$$F^h(X^c + X^h(X^c, \lambda), \lambda) = 0.$$

Thus, we may plug these solutions into (3.18a). The equation that remains to be solved is

$$r(X^c, \lambda) = F^c(X^c + X^h(X^c, \lambda), \lambda) = 0. \tag{3.19}$$

This completes the reduction as $r\colon \ker_0(L) \times \Lambda \to \ker_0(L)$ denotes a bifurcation problem on the generalized kernel. Equation (3.19) is also called the bifurcation equation. Since the subspaces are invariant under the monoid representation and the projections involved are equivariant, the same holds for r

$$r(T_\theta X_c, \lambda) = T_\theta r(X_c, \lambda)$$

for all $(X_c, \lambda) \in \ker_0(L) \times \Lambda$ (see Lemma 5.1 in Rink and Sanders [91]). Hence, we have reduced the bifurcation problem to an equivalent equivariant bifurcation problem on the generalized kernel. Due to the application of the implicit function theorem this describes solutions for the original problem only locally in an open neighborhood of (X_0, λ_0).

The procedure presented here is standard – apart from the slight adaptation to generalized kernel and reduced image. Our presentation follows Section 5 in Rink and Sanders [91]. This reduction technique allows to classify all possible branching steady state solutions that are generic in fundamental networks in terms of symmetries. As long as one has some information about possible generalized kernels it suffices to only investigate generic monoid equivariant systems on them. Note that this does not provide the full bifurcating branches – the $\operatorname{im}_0(L)$-component is missing. However, branching steady state solutions emerge from (X_0, λ_0) tangentially to $\ker_0(L)$ in the direction determined by solving the reduced equivariant bifurcation problem on the $\ker_0(L)$. Lyapunov-Schmidt reduction has been used in Rink and Sanders [91] to classify steady state bifurcations in all fundamental networks with 2 or 3 cells.

3.5 Center manifold reduction

3.5.1 The reduction method in generic bifurcation problems

The next reduction method we introduce provides a more complete picture but is also technically more involved. It allows to determine more information about bifurcating solutions than the Lyapunov-Schmidt reduction – for example stability of steady states – as well as the investigation of Hopf bifurcations. In general a dynamical system with a steady state at which the linearization has critical eigenvalues exhibits a – not necessarily unique – manifold, called *center manifold*, that is invariant under the dynamics and contains all solutions that are bounded. In particular, this includes all steady state and periodic solutions that emerge in a bifurcation problem. Hence, it suffices to restrict the investigation of local bifurcations to the center manifold. This technique is well-established in general dynamical systems as well as in (group) equivariant dynamics. However, it is not well-behaved with respect to network structure. More precisely, the dependencies of cells according to the interaction structure that is encoded in admissible vector fields can not be found in the restriction in general. One exception is known: in GANDHI et al. [40] and GOLUBITSKY and POSTLETHWAITE [46] the center manifold was shown to respect very simple feedforward structure. Once again, the equivalence of fundamental network structure and monoid symmetry (3.13) provides a means to encode structural features such that they are respected by the reduction method. Contrary to the Lyapunov-Schmidt reduction, however, this is not possible for arbitrary monoid equivariant dynamics. In order to guarantee existence of the center manifold while preserving (monoid) symmetry manipulations of the response functions are necessary. Hence, it depends crucially on the specific representation that is equivalent to the fundamental network structure. We introduce the technique similarly to the presentation in NIJHOLT, RINK, and SANDERS [79, 81] and highlight implications for general semigroup networks (not necessarily fundamental networks) in Remark 3.33 at the end of this subsection.

We use the notation introduced in Sections 3.2 and 3.4. Consider a smooth bifurcation problem driven by Γ_f, where $f \in \mathcal{C}^\infty(\bigoplus_{\sigma\in\Sigma} V \times \Lambda, V)$. Furthermore, without loss of generality, assume the bifurcation to occur at $(0,0) \in \bigoplus_{\sigma\in\Sigma} V \times \Lambda$. In particular, $\Gamma_f(0,0) = 0$ and $L = D_X\Gamma_f(0,0)$ has eigenvalues on the imaginary axis. This induces a splitting

$$\bigoplus_{\sigma\in\Sigma} V = \mathcal{X}^c \oplus \mathcal{X}^h,$$

where

$$\mathcal{X}^c = \bigoplus_{\mu\in\operatorname{spec}(L)\cap\mathbf{i}\mathbb{R}} \mathbb{E}_\mu, \quad \mathcal{X}^h = \bigoplus_{\mu\in\operatorname{spec}(L)\setminus\mathbf{i}\mathbb{R}} \mathbb{E}_\mu$$

are *center subspace* and *hyperbolic subspace* respectively. These subspaces are left invariant under the monoid representation

$$A_\sigma\mathcal{X}^c \subset \mathcal{X}^c, \quad A_\sigma\mathcal{X}^h \subset \mathcal{X}^h,$$

for all $\sigma \in \Sigma$. Once again we denote the projections along $\mathcal{X}^h$ and $\mathcal{X}^c$ respectively by

$$P^c\colon \bigoplus_{\sigma\in\Sigma} V \to \mathcal{X}^c, \quad P^h\colon \bigoplus_{\sigma\in\Sigma} V \to \mathcal{X}^h,$$

which are equivariant. These allow to uniquely split any element $X \in \bigoplus_{\sigma\in\Sigma} V$ as $X = X^c + X^h$ with $X^c = P^c(X)$ and $X^h = P^h(X)$.

Center manifolds in general are not explicitly constructed for bifurcation problems but for vector fields that are not parameter dependent. They can, however, easily be applied to the bifurcation setting by extending the system so as to include the parameter as a dynamic variable that does not evolve in time. To that end, choose coordinates $\underline{X} = (X,\lambda) \in \bigoplus_{\sigma\in\Sigma} V \times \Lambda$ and redefine the dynamical system to

$$\dot{\underline{X}} = \begin{pmatrix} \dot X \\ \dot\lambda \end{pmatrix} = \begin{pmatrix} \Gamma_f(X,\lambda) \\ 0 \end{pmatrix} = \underline{\Gamma}_f(\underline{X}). \tag{3.20}$$

Solutions of this system are in one-to-one correspondence with solutions of the parameter dependent system. In particular $\underline{\Gamma}_f(0,0) = (0,0)$. If we extend the representation matrices accordingly

$$\underline{A}_\sigma\colon \underline{X} = (X,\lambda) \mapsto (A_\sigma X, \lambda),$$

this system, furthermore, remains equivariant $\underline{\Gamma}_f \circ \underline{A}_\sigma = \underline{A}_\sigma \circ \underline{\Gamma}_f$.

As mentioned before, without loss of generality, we may consider $\Lambda = \mathbb{R}^l$, as we only investigate bifurcating solutions locally, i.e. for small values of λ. One readily computes

$$D\underline{\Gamma}_f(0,0) = D_{(X,\lambda)}\underline{\Gamma}_f(0,0) = \begin{pmatrix} D_X\Gamma_f(0,0) & D_\lambda\Gamma_f(0,0) \\ 0 & 0 \end{pmatrix}.$$

Hence, $D\underline{\Gamma}_f(0,0)$ has the same eigenvalues as $D_X\Gamma_f(0,0)$ joined by l times the additional eigenvalues 0 stemming from the λ-directions. Under the bifurcation assumption – eigenvalue 0 or purely imaginary eigenvalues – the linearization induces a splitting

$$\bigoplus_{\sigma\in\Sigma} V \times \mathbb{R}^l = \underline{\mathcal{X}}^c \oplus \underline{\mathcal{X}}^h$$

into center subspace and its complement of $D\underline{\Gamma}_f(0,0)$. These are invariant under the extended representation as well. Clearly, they are also related to the center and hyperbolic subspaces of the non-extended system. In particular,

$$\begin{aligned} \underline{\mathcal{X}}^h &= \mathcal{X}^h \times \{0\}, \\ \underline{\mathcal{X}}^c &= \mathcal{X}^c \times \{0\} \oplus \left\langle (X_1^h, \lambda_1), \dots, (X_l^h, \lambda_l) \right\rangle, \end{aligned} \tag{3.21}$$

where $\{\lambda_1, \dots, \lambda_l\}$ is a basis of $\mathbb{R}^l$ and $X_1^h, \dots, X_l^h \in \mathcal{X}^h$ are chosen suitably (see Lemma 5.3 in Nijholt, Rink, and Sanders [81]). Summarizing, the extended system has the same hyperbolic subspace as the non-extended system – with a trivial parameter component. The center subspace has additional spatial directions skewed with the parameter directions. The extended projections along $\underline{\mathcal{X}}^c$ and along $\underline{\mathcal{X}}^h$ respectively are denoted accordingly

$$\underline{P}^h \colon \bigoplus_{\sigma\in\Sigma} V \times \mathbb{R}^l \to \underline{\mathcal{X}}^h, \quad \underline{P}^c \colon \bigoplus_{\sigma\in\Sigma} V \times \mathbb{R}^l \to \underline{\mathcal{X}}^c,$$

and allow to uniquely split every $\underline{X} \in \bigoplus_{\sigma\in\Sigma} V \times \mathbb{R}^l$ as $\underline{X} = \underline{X}^c + \underline{X}^h$ with $\underline{X}^c = \underline{P}^c(\underline{X})$ and $\underline{X}^h = \underline{P}^h(\underline{X})$.

For later use, we define another projection that 'forgets' about the skewed directions of the extended center subspace. Note that every element in $\mathbb{R}^l$ can uniquely be represented in terms of its basis as $\lambda = \alpha_1\lambda_1 + \dots + \alpha_l\lambda_l$ with coefficients $\alpha_1, \dots, \alpha_l \in \mathbb{R}$. Thus, every element in $\underline{\mathcal{X}}^c$ has a unique representation as

$$\underline{X}^c = (X^c, 0) + \alpha_1(X_1^h, \lambda_1) + \dots + \alpha_l(X_l^h, \lambda_l)$$

where $X^c = P^c(X) \in \mathcal{X}^c$. Hence, the map

$$\begin{aligned} P' \colon \underline{\mathcal{X}}^c &\to \mathcal{X}^c \times \mathbb{R}^l \\ \underline{X}^c &\mapsto (X^c, \alpha_1\lambda_1 + \dots + \alpha_l\lambda_l) = (X^c, \lambda) \end{aligned} \tag{3.22}$$

is a projection. Note that this map satisfies $P' = (P^c \times \mathbb{1}_\Lambda)|_{\underline{\mathcal{X}}^c}$.

After restriction to a suitably small neighborhood around the bifurcation point $(0,0)$ – this involves restricting to Λ instead of $\mathbb{R}^l$ again – the extended system (3.20) admits a unique center manifold M^c, invariant under the dynamics, which locally contains all solutions whose $\underline{\mathcal{X}}^h$ component is bounded.

Theorem 3.32 (Theorems 4.1 and 5.1 in [81]). *Under the assumptions made above, there is a $\mathcal{C}^k$-function $\underline{\psi} \colon \underline{\mathcal{X}}^c \to \underline{\mathcal{X}}^h$ for any $k \in \mathbb{N}$ such that its graph is a manifold that is invariant under the dynamics induced by $\underline{\Gamma}_f$. More precisely, it contains all solutions with a bounded $\underline{\mathcal{X}}^h$-component. That is*

$$M^c = \{\underline{X}^c + \underline{\psi}(\underline{X}^c) \mid \underline{X}^c \in \underline{\mathcal{X}}^c\} = \left\{ \underline{X} \in \bigoplus_{\sigma\in\Sigma} V \times \Lambda \;\middle|\; \sup_{t\in\mathbb{R}} \|\underline{P}^h\phi^t(\underline{X})\| < \infty \right\}, \tag{3.23}$$

where ϕ^t is the flow induced by $\underline{\Gamma}_f$ and $\|\cdot\|$ denotes the Euclidean norm. The function $\underline{\psi}$ satisfies $\underline{\psi}(0,0) = 0$ and $D\underline{\psi}(0,0) = 0$ so that the manifold is tangential to the center subspace in the bifurcation point. Furthermore, $\underline{\psi} \circ \underline{A}_\sigma = \underline{A}_\sigma \circ \underline{\psi}$ for all $\sigma \in \Sigma$, where the representation is restricted to the corresponding subspaces, which yields that the center manifold is invariant under the monoid representation, $\underline{A}_\sigma(X,\lambda) \in M^c$ for all $(X,\lambda) \in M^c, \sigma \in \Sigma$.

The projection $P = P' \circ \underline{P}^c \colon \bigoplus_{\sigma\in\Sigma} V \times \Lambda \to \mathcal{X}^c \times \Lambda$ with $P(X,\lambda) = (P^c(X),\lambda)$[4] is equivariant as well: $P \circ \underline{A}_\sigma = \underline{A}_\sigma \circ P$ for all $\sigma \in \Sigma$. Its restriction to the center manifold $P|_{M^c}$ bijectively conjugates the flow on the center manifold induced by $\underline{\Gamma}_f\big|_{M^c}$ to that induced by a system on $\mathcal{X}^c \times \Lambda$ given by an ordinary differential equation of the form

$$\begin{aligned} \dot{X}^c &= r(X^c,\lambda), \\ \dot{\lambda} &= 0. \end{aligned} \tag{3.24}$$

That is, $r\colon \mathcal{X}^c \times \Lambda \to \mathcal{X}^c$ describes the reduced *bifurcation problem on the center subspace of the linearization and has the following properties:*

(i) $r(0,0) = 0$.

(ii) The center subspace of $D_{X^c}r(0,0)$ is the full space $\mathcal{X}^c$.

(iii) It is equivariant with respect to the monoid representation restricted to $\mathcal{X}^c$

$$r(A_\sigma X^c,\lambda) = A_\sigma r(X^c,\lambda)$$

for all $\sigma \in \Sigma$ and $(X^c,\lambda) \in \mathcal{X}^c \times \Lambda$.

The center manifold theorem consists of two parts. The first describes the existence of the center manifold (3.23) and its properties. It is invariant under the dynamics as well as under the symmetries and most importantly contains all the 'interesting' dynamics. The second part describes the actual reduction method. The dynamics restricted to the center manifold are equivalent to those of a parameter dependent system on the center subspace of the non-extended system $\mathcal{X}^c$ (3.24). As the center manifold contains all the dynamical features we are interested in – in particular branches of steady state and periodic solutions as well as their stability information – the same is to be said about the reduced system. Hence, this is a powerful tool to investigate bifurcations in a fixed bifurcation problem – that is, for a fixed response function f. Compared to the Lyapunov-Schmidt reduction, it provides more information. This, however, comes at the cost of increased computational difficulty. In general, one cannot determine the center manifold exactly. There are methods to compute Taylor coefficients of the function $\underline{\psi}$ whose graph is the center manifold. Then, after a truncation at some degree, one needs to work with an approximation. This, however, is usually true for the Lyapunov-Schmidt reduction as well, since the computation of the reduced equation (3.19) requires application of the implicit function theorem.

Interestingly, both reduction methods yield similar methods to classify *generic* bifurcations in fundamental networks. The generic reduced bifurcation problems (3.19) and (3.24) are the same when investigating steady state bifurcations. Lyapunov-Schmidt reduction can also be applied to general monoid equivariant bifurcation problems. The center manifold reduction, on the other hand, guarantees that more information can be gained. Furthermore, it also applies to Hopf bifurcations. As a matter of fact, any parameter dependent vector field on the center subspace $g\colon \mathcal{X}^c \times \Lambda \to \mathcal{X}^c$ satisfying points (i)-(iii) – bifurcation point $(0,0)$, only critical eigenvalues, equivariance – in the second part of Theorem 3.32 can be obtained as a reduced vector field r of an extended fundamental network vector field $\underline{\Gamma}_f$ (Theorems 5.4 and 5.5 in Nijholt, Rink, and Sanders [81]). Hence, determining all these vector fields and classifying their generic bifurcations also gives a classification of bifurcations for the fundamental network vector fields.

[4] This projection satisfies $P = (P^c \times \mathbb{1}_\Lambda)$.

Remark 3.33. The center manifold reduction cannot be applied directly to non-fundamental networks. Nevertheless, it provides information about the dynamics of an original network via a detour through its fundamental network. Recall that the relation between these two networks is given by input networks and robust patterns of synchrony (Corollary 3.25). That is, the dynamics of the input network of a cell $p \in C$ of the original network is found in the robust synchrony subspace $\Delta_p \subset \bigoplus_{\sigma\in\Sigma} V$ of the fundamental network. As a matter of fact, the center manifold reduction method respects robust synchrony subspaces in the sense that

$$P\colon \{(X,\lambda) \in M^c \mid X \in \Delta_p, \lambda = \overline{\lambda}\} \to \{(X^c,\lambda) \in \mathcal{X}^c \times \Lambda \mid X^c \in \Delta_p, \lambda = \overline{\lambda}\},$$

is a bijection for every $\overline{\lambda} \in \Lambda$, where P is the projection from Theorem 3.32. Hence, restricting the reduced system to the synchrony subspace provides a reduced system for the input network. Furthermore, as a generic reduced system on $\mathcal{X}^c$ determines the dynamics/bifurcations of a fundamental network for a generic response function f, this restriction also preserves genericity, the response function is also generic for the input network. In particular, when we are in the situation that the original network is contained in its fundamental network as a quotient, the generic bifurcations are entirely determined by the restriction of the reduced system of its fundamental network to the corresponding synchrony subspace. △

So far, we have avoided to address the question how to determine which subspaces of $\bigoplus_{\sigma\in\Sigma} V$ are possible as center subspaces $\mathcal{X}^c$ to begin with. As we have mentioned before, restrictions induced by network structure or equivalently by symmetry have a strong influence on spectral properties of the linearizations of admissible vector fields. As a result, a generic fundamental network vector field does not allow for an arbitrary subspace as its center subspace. There are, however, ways to characterize possible center subspaces in terms of representation theory (Chapter 4), which will be the topic of Chapter 5.

3.5.2 Reduction by projection blocks

Finally, let us mention that the center manifold can also be used as a powerful theoretical tool to prove results on (generic) bifurcations. Its existence and the fact that it contains all branching solutions can be more important than its actual computation in order to derive general statements about bifurcations of networks. One example is the reduction by so-called projection blocks that was introduced in NIJHOLT, RINK, and SANDERS [80]. This method allows to investigate some generically occurring bifurcations of a semigroup network in terms of specific quotient networks. The suitable quotients are determined purely combinatorically.

Definition 3.34 (Definition 6.1 in [80]). *A subset $B \subset C$ of the cells of $\mathcal{N}$ is called a* block *if there are no arrows that start at cells outside of B pointing at cells inside of B. In terms of the input maps, this means $\sigma(B) \subset B$ for all $\sigma \in \Sigma$. A block is a projection block, if additionally there exists an input map $\kappa \in \Sigma$ such that $\kappa(C) = B$ and $\kappa(B) = B$.*

Clearly, not every network contains a projection block. The full network, however, is always an example of a block. Note that each block $B \subset C$ gives rise to a balanced partition. That is, $P = \{B, \{p_1\}, \dots, \{p_k\}\}$, where $C \setminus B = \{p_1, \dots, p_k\}$, is balanced. Each of the sets in P gets mapped into only one subset due to the definition of a block, hence, satisfying the condition in Proposition 3.11. It induces the quotient network $\mathcal{N}_B$ in which B is identified with a single cell that has a self-loop of every color. Its total phase space is contained in that of the original network as the robust synchrony subspace

$$\Delta_B = \Delta_P = \left\{ x \in \bigoplus_{p\in C} V \;\middle|\; x_p = x_q, \quad \text{if} \quad p,q \in B \right\}.$$

In Remark 3.33 we describe how the center manifold reduction behaves in regard to the restriction to synchrony subspaces at the example of a network $\mathcal{N}$ that is contained in its fundamental network $\widetilde{\mathcal{N}}$ as a quotient. We can determine generic steady state bifurcations of $\mathcal{N}$ from reduced vector fields on its possible center subspaces. Furthermore, the center manifold of the fundamental network restricts to one of the original network. Hence, generic bifurcations of the fundamental network restricted to the corresponding synchrony subspace are also generic for its quotient. Interestingly, given a network containing a projection block (which is contained in its fundamental network as a quotient) this propagation of generic behavior can be reversed. As a matter of fact, any generic center subspace for an extended system of $\mathcal{N}_B$ is also a generic center manifold for the original network $\mathcal{N}$ when considered as a submanifold of $\Delta_B \times \mathbb{R}^l \subset \bigoplus_{p\in C} V \times \mathbb{R}^l$. More precisely, for a generic bifurcation problem of the quotient $\mathcal{N}_B$ governed by the smooth parameter dependent vector field $\hat{\gamma}_g$ – i.e. $\hat{\gamma}_g(0,0) = 0$ and $\operatorname{spec}(D_x\hat{\gamma}_g(0,0))$ contains critical eigenvalues – let γ_f be a generic vector field such that $\gamma_f|_{\Delta_B\times\Lambda} = \hat{\gamma}_g$. Then the locally defined center manifold for $\hat{\gamma}_g$ is also a local center manifold for γ_f (both in the sense of Remark 3.33). In particular, as laid out in the previous subsection, the bifurcations in these generic systems agree. Furthermore, any admissible vector field for $\mathcal{N}$ can be restricted to one for $\mathcal{N}_B$ via the synchrony subspace and any network vector field for $\mathcal{N}_B$ can be realized by such a restriction. Hence, generic behavior for $\mathcal{N}_B$ is also generic for $\mathcal{N}$ (compare to Corollary 6.8 in Nijholt, Rink, and Sanders [80]).

Note that this method does not necessarily allow to describe *all* generic bifurcations of the original network $\mathcal{N}$ for two reasons. First, if a generic bifurcation problem governed by γ_f yields a center subspace that has a trivial intersection with the synchrony subspace corresponding to the block Δ_B the quotient network cannot provide any information on the bifurcations that arise. In this setting branching occurs in directions that are transversal to the synchrony subspace Δ_B. Secondly, only specific generic bifurcations of $\mathcal{N}_B$ are also generic for $\mathcal{N}$. In particular, those for which the center subspace of $D_x\hat{\gamma}_f(0,0)$ has a trivial intersection with the fully synchronous subspace are suitable.

Nevertheless, the technique shows an interesting application of the center manifold reduction as a theoretical tool. It allows to describe generic bifurcations by only investigating a suitable quotient network. Another example can be seen in Chapter 11, where we make heavy use of center manifolds to prove that certain networks exhibit the same bifurcation patterns independent of their internal phase space. To that end, we include some additional technical results in the upcoming subsection.

Remark 3.35. The reduction by projection blocks addresses what is also known as the *lifting bifurcation problem* even though it was not developed in this regard. Some research – not restricted to semigroup networks and their fundamental networks – has been devoted to the question when the bifurcation behavior of a network is different from that of a quotient. More precisely, given two networks $\mathcal{N}, \mathcal{N}'$ such that $\mathcal{N}$ is a quotient of $\mathcal{N}'$ by a balanced partition. Then $\mathcal{N}'$ is called a *lift* of $\mathcal{N}$. Any bifurcating branch of solutions for a system on $\mathcal{N}$ also describes a branch of solutions for a system on $\mathcal{N}'$ with the same center subspace via restriction to the synchrony subspace. The question remains, whether additional branches can occur from a bifurcation in the lift with the same center subspace – in particular, branches in which certain cells are not synchronous any longer. The result in this section gives a partial answer for the restricted case of semigroup networks and quotients by projection blocks. Generically, the bifurcations observed for the quotient *lift* to the full network as described above and no additional branches are possible (see Corollary 6.8 in Nijholt, Rink, and Sanders [80]). Only when the original network undergoes a bifurcation through a center subspace that is not compatible to the quotient by a projection block, new bifurcations can be observed. For more information on the lifting bifurcation problem in different cases see for example Aguiar et al. [8], Moreira [74], and Soares [99] and the references therein, without a claim of this list being complete. △

3.5.3 Additional technicalities

For the sake of completeness and especially for later use, we include some additional technicalities and state some results on the center manifold for fundamental networks that were not explicitly provided in NIJHOLT, RINK, and SANDERS [81]. First of all we describe the impact of equivariance with respect to the extended symmetries on an arbitrary map in the following lemma.

Lemma 3.36. *Let*

$$F\colon \bigoplus_{\sigma\in\Sigma} V\times\mathbb{R}^l \to \bigoplus_{\sigma\in\Sigma} V\times\mathbb{R}^l$$
$$(X,\lambda)\mapsto \begin{pmatrix} F_X(X,\lambda)\\ F_\lambda(X,\lambda)\end{pmatrix}$$

be equivariant with respect to the extended monoid representation: $F\circ\underline{A}_\sigma=\underline{A}_\sigma\circ F$ *for all* $\sigma\in\Sigma$*. Then the component functions* F_X *and* F_λ *are parameter dependent maps, which are equivariant and invariant with respect to the non-extended monoid representation respectively. That is,*

$$F_X(A_\sigma X,\lambda)=A_\sigma F_X(X,\lambda),\qquad F_\lambda(A_\sigma X,\lambda)=F_\lambda(X,\lambda).$$

Proof. This can be seen directly from the equivariance condition

$$\left(F\circ\underline{A}_\sigma\right)(X,\lambda)=\begin{pmatrix} F_X(A_\sigma X,\lambda)\\ F_\lambda(A_\sigma X,\lambda)\end{pmatrix}=\begin{pmatrix} A_\sigma F_X(X,\lambda)\\ F_\lambda(X,\lambda)\end{pmatrix}=\left(\underline{A}_\sigma\circ F\right)(X,\lambda).$$

□

Then, we prove two technical results that explicitly describe the parameter-dependence of the center manifold of the extended system M^c and the maps that define the interconnection between center manifold and center subspace $\underline{\mathcal{X}}^c$. They can be summarized by stating that the center manifold and the center subspace of the extended system share the same parameter component.

Lemma 3.37 (see also the proof of Lemma 5.3 in [81])**.** *Let*

$$\underline{P}^c\colon \bigoplus_{\sigma\in\Sigma} V\times\mathbb{R}^l\to\underline{\mathcal{X}}^c \tag{3.25}$$

be the equivariant projection onto the center subspace along the hyperbolic subspace. Then $\underline{P}^c$ *acts as the identity on the* λ*-component, i.e.* $\underline{P}^c(X,\lambda)=(\underline{P}^c_X(X,\lambda),\lambda)$*.*

Proof. Fix $(X,\lambda)\in\bigoplus_{\sigma\in\Sigma}V\times\mathbb{R}^l$. Recall that

$$\bigoplus_{\sigma\in\Sigma}V=\mathcal{X}^c\oplus\mathcal{X}^h.$$

Hence, we find a unique representation $X=X^c+X^h$ with $X^c\in\mathcal{X}^c$ and $X^h\in\mathcal{X}^h$ using projections $X^c=P^c(X), X^h=P^h(X)$. From the representation in (3.21), we see that $(X^c,0)\in\underline{\mathcal{X}}^c$. Furthermore, we may represent the parameter as $\lambda=\alpha_1\lambda_1+\cdots+\alpha_l\lambda_l$, where $\alpha_1,\ldots,\alpha_l\in\mathbb{R}$, and it holds that

$$\alpha_1(X_1^h,\lambda_1)+\cdots+\alpha_l(X_l^h,\lambda_l)\in\underline{\mathcal{X}}^c.$$

Lastly, as $X_1^h,\ldots,X_l^h\in\mathcal{X}^h$ we obtain

$$\left(X^h-(\alpha_1X_1^h+\cdots+\alpha_lX_l^h),0\right)\in\underline{\mathcal{X}}^h.$$

Since $\underline{P}^c$ is the projection onto $\underline{\mathcal{X}}^c$, it acts as the identity on $\underline{\mathcal{X}}^c$ and maps $\underline{\mathcal{X}}^h$ to 0. We obtain

$$\begin{aligned}\underline{P}^c\colon (X,\lambda) &= (X^c + X^h, \lambda)\\ &= (X^c,0) + (X^h - (\alpha_1 X_1^h + \cdots + \alpha_l X_l^h), 0) + \alpha_1(X_1^h,\lambda_1) + \cdots + \alpha_l(X_l^h,\lambda_l)\\ &\mapsto (X^c,0) + 0 + \alpha_1(X_1^h,\lambda_1) + \cdots + \alpha_l(X_l^h,\lambda_l)\\ &= \left(X^c + \alpha_1 X_1^h + \cdots + \alpha_l X_l^h, \lambda\right).\end{aligned}$$

□

Lemma 3.38. *Let $\underline{\psi}\colon \underline{\mathcal{X}}^c \to \underline{\mathcal{X}}^h$ be the map, whose graph is the center manifold M^c as in* (3.23). *Then $\underline{\psi}$ has a trivial λ-component, i.e. $\underline{\psi}(X^c,\lambda) = \left(\underline{\psi}_X(X,\lambda), 0\right)$.*

Proof. This follows directly from the representation of the subspaces given in (3.21) $\underline{\mathcal{X}}^h = \mathcal{X}^h \times \{0\}$. □

Finally, note that we have not made the conjugacy between the system on the center manifold M^c and the reduced system on $\mathcal{X}^c \times \Lambda$ precise. In particular, so far we have only provided a map $P\colon M^c \to \mathcal{X}^c \times \Lambda$. The next two results provide its inverse.

Lemma 3.39 (see also the proof of Theorem 5.4 in [81])**.** *The projection $P'\colon \underline{\mathcal{X}}^c \to \mathcal{X}^c \times \mathbb{R}^l$ is invertible with inverse given by*

$$\begin{aligned} Q'\colon \mathcal{X}^c \times \mathbb{R}^l &\to \underline{\mathcal{X}}^c\\ (X^c,\lambda) &\mapsto (X^c,0) + \alpha_1(X_1^h,\lambda_1) + \cdots + \alpha_l(X_l^h,\lambda_l) = \underline{X}^c,\end{aligned} \tag{3.26}$$

where $\lambda = \alpha_1\lambda_1 + \cdots + \alpha_l\lambda_l$ is a representation in the basis $\mathbb{R}^l = \langle \lambda_1,\ldots,\lambda_l\rangle$ and $X_1^h,\ldots,X_l^h \in \mathcal{X}^h$ suitable as in (3.21).

Proof. This follows directly from the definition of P' in (3.22). □

Corollary 3.40. *We can characterize the center manifold as a graph over $\mathcal{X}^c \times \Lambda$ using the map Q' from Lemma 3.39:*

$$M^c = \left\{ Q'(X^c,\lambda) + \underline{\psi}(Q'(X^c,\lambda)) \mid (X^c,\lambda) \in \mathcal{X}^c \times \Lambda \right\}.$$

Then we have $Q'(0,0) = 0$, $(\underline{\psi} \circ Q')(0,0) = 0$ and $D(\underline{\psi} \circ Q')(0,0) = 0$. In particular, $\underline{\psi} \circ Q'$ is inverse to the map P from Theorem 3.32 restricted to M^c.

3.6 Non-homogeneous networks

In this final section about semigroup networks, we describe how much of the theory explained above can be generalized to *non-homogeneous* networks. That is, we allow different *types* or *shapes* of cells. As we mentioned before, homogeneity is not crucial. The assumption of asymmetric inputs – only one input of each color into a cell –, however, cannot be dropped. The algebraic structure that encodes these types of networks is that of a *semigroupoid* – i.e. a semigroup in which not all products are defined, informally speaking. Most of the general framework has straightforward generalizations to this case. The investigation of dynamics, on the other hand, is a lot more involved. This is due to the fact that semigroupoid representations are not as well behaved. Since the notation needed is a lot heavier and since we do not pursue this generalization any further in this thesis, we do not give any details but rather provide a brief overview. The semigroupoid approach to non-homogeneous networks has been proposed in NIJHOLT, RINK, and SANDERS [78] and RINK and SANDERS [92] without any proofs of the main results. These are mostly direct adaptations of corresponding results for the homogeneous case. Furthermore, they can be found in NOETHEN [85].

In order to realize multiple types of cells in the framework, one introduces a partition of the set of cells according to their types and adapts all definitions in terms of the consistency conditions as given for example in the groupoid formalism. In particular, all arrows of a given color go from a cell of type A to a cell of type B. Under the assumptions of asymmetric inputs, these can again be encoded in terms of input maps that are restricted to suitable cell types (mapping only cells of type B to cells of type A). Then, an internal phase space and a response function is attached to each cell which coincide for cells of the same type. We obtain admissible vector fields and coupled cell systems in just the same way as before.

Furthermore, we may again concatenate input maps. This concatenation has the same interpretation as before: finding indirect inputs by first following back an arrow of color 1 followed by an arrow of color 2. However, not all concatenations are possible. More precisely, the types of heads and tails have to be compatible: the input map corresponding to color 2 needs to describe arrows into cells of the type from which arrows of color 1 emerge. Constructing the closure under concatenations – i.e. including indirect inputs that emerge in this manner as direct inputs – gives the set of input maps the structure of a semigroupoid. Accordingly, we call a network whose set of input maps is a semigroupoid a *semigroupoid network*. Some of the basic structural features can be determined in just the same way as in the homogeneous case, i.e. symmetries, graph fibrations, and robust synchrony patterns are subject to the (adapted) conditions of Definitions 3.10 and 3.12 and Proposition 3.11 respectively.

We may also construct fundamental networks for semigroupoid networks. The consistency condition and the fact that not all input maps can be multiplied yields a collection of fundamental networks, one for each cell type of the original network. Their input structures are defined similar to the ones before, where special care needs to be taken as to which input maps are contained as cells in one fundamental network and which can act as input maps thereon. Nevertheless, each cell induces a graph fibration from the fundamental network corresponding to its type to the original network as in Theorem 3.24. Hence, once again input networks are contained in fundamental networks as quotients by a balanced partition.

The equivariance of fundamental network vector fields has its counterpart for non-homogeneous networks as well. For this to make sense, we have to consider all fundamental networks collectively. Every input map gives rise to a graph fibration from one fundamental network to another according to the rules of concatenation. These, in turn, induce linear semiconjugacies between the corresponding admissible vector fields, i.e. they map the total phase space of one fundamental network to that of another and intertwine the fundamental network vector fields thereon as in Theorem 3.26. Furthermore, these linear semigconjugacies respect the rules of multiplication in the semigroupoid. Hence, they can be interpreted as a representation of the semigroupoid of input maps on the collection of total phase spaces. A collection of admissible vector fields on these total phase spaces is then in some sense equivariant – semigroupoid-equivariant – with respect to this representation if the individual functions are intertwined as described above. As before, this generalized equivariance condition is actually equivalent to admissibility for the collection of fundamental networks. To our knowledge this has only been noted in Noethen [85].

Even though fundamental network vector fields can be characterized entirely in terms of generalized symmetries, it is a lot harder to exploit this property for the investigation of dynamics than in homogeneous networks. Admissible vector fields and graph fibrations can be generalized quite directly. Hence, dynamics as well as the implications of the fundamental network structure may be investigated in ways similar to the homogeneous case. However, in particular some of the reduction methods introduced in Sections 3.3 to 3.5 pose serious challenges. The semigroupoid extension introduces a Lie algebra structure on the admissible vector fields by a straightforward generalization of the construction for homogeneous networks. This facilitates a normal form reduction for non-homogeneous networks as in Theorem 3.31. The other reduction methods – Lyapunov-Schmidt reduction and center manifold reduction –, however, rely crucially on the fact that symmetries of the fundamental network can be reduced to generalized kernels and center subspaces, when these are complemented by an invariant subspace themselves. At the moment it is not clear, how this generalizes to the even more general symmetries induced by semigroupoid networks. In particular, currently there is no generalization of the results in Sections 3.4 and 3.5 to non-homogeneous networks.

Chapter 4

Representation theory of monoids

In this chapter we provide background and establish notation for the representation theory of monoids. We include only chosen basic results that we require for the upcoming investigations. The full picture on monoid representation theory is far beyond the scope of this thesis. For more information we refer to STEINBERG [105]. This beautiful book aims at making monoid representation theory accessible to non-experts who wish to apply it in their work and has deeply inspired our research. For the most part we follow our presentation in SCHWENKER [97] to which we add some additional results and remarks. We omit most of the proofs of the results in Section 4.1, as they are neatly presented in RINK and SANDERS [91]. Some further additions can be found in NIJHOLT and RINK [77].

We describe representations of a monoid on finite-dimensional real vector spaces. However, we do not impose any further restrictions on the monoid. It may be finite or infinite and, therefore, includes finite or compact symmetry groups that form the classical symmetries in equivariant dynamics. But also non-compact groups are included in this framework. The proofs in RINK and SANDERS [91] are stated for finite monoids. However, only the fact that the representations are finite-dimensional is used, and therefore they also apply in the more general case. The right regular representation that encodes the fundamental network structure of a semigroup network is one example to which we want to apply the derived results. In Section 4.2 we take an abstractly algebraic look into the representation theory involved.

4.1 The basics

Definition 4.1. *The tuple* $(\Theta, \cdot)$*, where* Θ *is a set and* $\cdot\colon \Theta \times \Theta \to \Theta$ *is a map so that*

- *(i)* $(\theta \cdot \theta') \cdot \tilde{\theta} = \theta \cdot (\theta' \cdot \tilde{\theta})$ *for all* $\theta, \theta', \tilde{\theta} \in \Theta$*, (associativity)*
- *(ii) there exists* $\mathrm{Id} \in \Theta$ *so that* $\mathrm{Id} \cdot \theta = \theta \cdot \mathrm{Id} = \theta$ *for all* $\theta \in \Theta$*, (neutral element)*

is called a monoid *with* multiplication $\cdot$*. We abbreviate* $\theta\theta' = \theta \cdot \theta'$ *and call* Θ *the monoid if the multiplication is clear from context.*

In particular, any group is an example of a monoid in which for every $\theta \in \Theta$ there is an element $\theta^{-1} \in \Theta$ such that $\theta\theta^{-1} = \theta^{-1}\theta = \mathrm{Id}$. The definition of a *representation* is well known in the context of groups and we define it accordingly for monoids. To that end, let $\mathbf{V}$ be a finite-dimensional real vector space and $\mathfrak{gl}(\mathbf{V})$ the real vector space of linear maps from $\mathbf{V}$ to itself. We abbreviate $LL' = L \circ L'$ for $L, L' \in \mathfrak{gl}(\mathbf{V})$ and interpret it as a multiplication, which gives it the structure of an algebra – or a monoid if we only consider multiplication. A map

$$T\colon \Theta \to \mathfrak{gl}(\mathbf{V}), \quad \theta \mapsto T_\theta,$$

with

(i) $T_\theta T_{\theta'} = T_{\theta\theta'}$ for all $\theta, \theta' \in \Theta$

(ii) $T_{\mathrm{Id}} = \mathbb{1}_{\mathbf{V}}$

is a *homomorphism of monoids* – $\{T_\theta \mid \theta \in \Theta\} \subset \mathfrak{gl}(\mathbf{V})$ is a submonoid. Any homomorphism $\Theta \to \mathfrak{gl}(\mathbf{V})$ defines a *representation* of Θ. When the homomorphism is known from context or not needed in its explicit form we also refer to $\mathbf{V}$ as the representation of Θ.

Consider a second representation $T' \colon \Theta \to \mathfrak{gl}(\mathbf{V}')$ on a finite-dimensional real vector space $\mathbf{V}'$ and a linear map $L \colon \mathbf{V} \to \mathbf{V}'$. If L intertwines the monoid representations

$$L \circ T_\theta = T'_\theta \circ L$$

for all $\theta \in \Theta$, we call it a *homomorphism of representations* and write

$$L \in \mathrm{Hom}_\Theta(\mathbf{V}, \mathbf{V}').$$

This set has the structure of a real vector space. If $\mathbf{V} = \mathbf{V}'$, we call L an *endomorphism* and write

$$\mathrm{End}_\Theta(\mathbf{V}) = \mathrm{Hom}_\Theta(\mathbf{V}, \mathbf{V}) = \{L \in \mathfrak{gl}(\mathbf{V}) \mid L \circ T_\theta = T_\theta \circ L \text{ for all } \theta \in \Theta\}.$$

As we may concatenate two endomorphisms, this has the structure of a real algebra. An invertible homomorphism of representations is called an *isomorphism*. If there is an isomorphism between $\mathbf{V}$ and $\mathbf{V}'$ we call these representations *equivalent* or *isomorphic* denoted by $\mathbf{V} \cong \mathbf{V}'$. The following remark points out why endomorphisms are especially important for the study of monoid equivariant dynamics.

Remark 4.2. Assume that $F \colon \mathbf{V} \to \mathbf{V}$ is a continuously differentiable vector field and $X_0 \in \mathbf{V}$ is a point such that

(i) X_0 is an equilibrium point of F, i.e., $F(X_0) = 0$;

(ii) X_0 is Θ-*symmetric*, i.e. $T_\theta X_0 = X_0$ for all $\theta \in \Theta$;

(iii) F is Θ-*equivariant*, i.e. $F \circ T_\theta = T_\theta \circ F$ for all $\theta \in \Theta$.

Then differentiation of $F(T_\theta X) = T_\theta F(X)$ at $X = X_0 = T_\theta X_0$ yields

$$L \circ T_\theta = T_\theta \circ L$$

with $L = DF(X_0)$ and hence $L \in \mathrm{End}_\Theta(\mathbf{V})$. This is particularly important for bifurcation analysis. Assume the vector field additionally depends on a parameter $\lambda \in \mathbb{R}^l$ and that $X_0(\lambda)$ denotes a smooth curve of Θ-symmetric equilibria, i.e. $F(X_0(\lambda), \lambda) = 0$ and $T_\theta X_0(\lambda) = X_0(\lambda)$ for all $\theta \in \Theta$ and for all $\lambda \in \mathbb{R}^l$. Then the linearizations

$$L_\lambda = D_X F(X_0(\lambda), \lambda) \in \mathrm{End}_\Theta(\mathbf{V})$$

describe a smooth family of endomorphisms of $\mathbf{V}$. △

Remark 4.3. A representation that is of particular interest to us is the right regular representation $\sigma \mapsto A_\sigma$ on $\bigoplus_{\sigma\in\Sigma} V$ given by $(A_\sigma(X))_\tau = X_{\tau\sigma}$ that is equivalent to the fundamental network structure. As was pointed out in Corollary 3.23 and Remark 3.29, the endomorphisms are spanned by the generalized adjacency matrices defined as $(B_\sigma(X))_\tau = X_{\sigma\tau}$. In particular, for every $L \in \mathrm{End}_\Sigma\left(\bigoplus_{\sigma\in\Sigma} V\right)$ there exist linear maps $b_\sigma \in \mathfrak{gl}(V)$ for every $\sigma \in \Sigma$ such that

$$(LX)_\tau = \sum_{\sigma\in\Sigma} b_\sigma (B_\sigma X)_\tau$$

as in Proposition 3.8. In particular, for $V = \mathbb{R}$, we identify $\mathfrak{gl}(\mathbb{R}) \cong \mathbb{R}$ and obtain

$$\mathrm{End}_\Sigma\left(\bigoplus_{\sigma\in\Sigma} \mathbb{R}\right) = \langle B_\sigma \mid \sigma \in \Sigma \rangle$$

as a finite-dimensional real algebra. △

A subspace $W \subset \mathbf{V}$ is called a *subrepresentation* if it is *invariant* under the action of the monoid:

$$T_\theta W \subset W$$

for all $\theta \in \Theta$. The representation $\mathbf{V}$ is called *irreducible* if there exists no proper subrepresentation $W \subset \mathbf{V}$ with $W \neq \{0\}, \mathbf{V}$. It is called *indecomposable* if there are no two proper subrepresentations W, W' with $\mathbf{V} = W \oplus W'$. Unlike for group representations, these two properties are not equivalent. An indecomposable representation need not be irreducible. More precisely, a nontrivial subrepresentation W of a group representation can always be complemented by another nontrivial subrepresentation. We say W is *complementable*. This is not true for monoid representations, however. Nevertheless, any monoid representation $\mathbf{V}$ decomposes into finitely many indecomposable subspaces

$$\mathbf{V} = W_1 \oplus \cdots \oplus W_m.$$

This decomposition is unique up to equivalence of subrepresentations, which is stated in the following theorem.

Theorem 4.4 (Krull-Schmidt theorem, Theorem 4.9 in [91]). *Let $\mathbf{V}$ be a finite-dimensional real representation of Θ and let*

$$\mathbf{V} = W_1 \oplus \cdots \oplus W_m$$

be a decomposition of $\mathbf{V}$ into indecomposable subrepresentations. Then this decomposition is unique up to isomorphisms. That is, if it also holds that

$$\mathbf{V} = W_1' \oplus \cdots \oplus W_{m'}'$$

with indecomposable subrepresentations $W_1', \dots, W_{m'}'$ then $m = m'$ and $W_i \cong W_i'$ for all i after renumbering the subrepresentations.

Note that endomorphisms give rise to multiple (possibly nontrivial) subrepresentations. For example, for $L \in \operatorname{End}_\Theta(\mathbf{V})$ its image and kernel are examples of subrepresentations of $\mathbf{V}$. More generally, for every eigenvalue $\mu \in \operatorname{spec}(L)$ the eigenspace E_μ is invariant under the representation which can easily be checked from the equivariance of L. In particular, the same holds for direct sums of eigenspaces. These subrepresentations are in general not complementable. The generalized eigenspaces $\mathbb{E}_\mu$ and their direct sums, on the other hand, are

$$\mathbf{V} = \mathbb{E}_\mu \oplus \bigoplus_{\mu' \in \operatorname{spec}(L) \setminus \{\mu\}} \mathbb{E}_{\mu'}.$$

Just as in Sections 3.4 and 3.5 we use this to define *generalized kernel* and *reduced image* as

$$\ker_0(L) = \mathbb{E}_0 \quad \text{and} \quad \operatorname{im}_0(L) = \bigoplus_{\mu \in \operatorname{spec}(L) \setminus \{0\}} \mathbb{E}_\mu \tag{4.1}$$

as well as *center subspace* and *hyperbolic subspace* as

$$\mathcal{X}^c = \bigoplus_{\mu \in \operatorname{spec}(L) \cap \mathbf{i}\mathbb{R}} \mathbb{E}_\mu \quad \text{and} \quad \mathcal{X}^h = \bigoplus_{\mu \in \operatorname{spec}(L) \setminus \mathbf{i}\mathbb{R}} \mathbb{E}_\mu. \tag{4.2}$$

This allows us to describe subspaces with crucial relevance for bifurcation analysis – via the reduction methods mentioned before – in terms of complementable subrepresentations.

For the next results we focus on indecomposable representations altogether. In that case, the algebra of endomorphisms has interesting properties itself.

Proposition 4.5 (Fitting Lemma, Proposition 4.4 in [91]). *Let $\mathbf{V}$ be an indecomposable representation and let $L \in \operatorname{End}_\Theta(\mathbf{V})$. Then L is either invertible or nilpotent (i.e., there exists $n \in \mathbb{N}$ such that $L^n = 0 \in \mathfrak{gl}(\mathbf{V})$).*

As a corollary we obtain that the set of nilpotent endomorphisms of an indecomposable monoid representation $\mathbf{V}$

$$\mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{V}) = \{L \in \mathrm{End}_\Theta(\mathbf{V}) \mid L \text{ is nilpotent}\}$$

is an ideal in $\mathrm{End}_\Theta(\mathbf{V})$. That is for $L \in \mathrm{End}_\Theta(\mathbf{V}), N, N' \in \mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{V})$ and $\mu \in \mathbb{R}$, we also have that $LN, NL, N + N'$ and μN are nilpotent. Hence, we may consider the quotient algebra by factoring out this ideal and obtain

Theorem 4.6 (Schur's lemma, Lemma 4.8 in [91])**.** *Let $\mathbf{V}$ be an indecomposable representation. The quotient*

$$\mathrm{End}_\Theta(\mathbf{V})/\,\mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{V}) \tag{4.3}$$

is a real division algebra.

Remark 4.7. Schur's Lemma originally stems from module theory where it describes endomorphisms of simple modules and is especially important in representation theory of groups. This formulation is a significant generalization to indecomposable modules. △

Recall that by the Frobenius theorem (see for example FAITH [34]) any finite-dimensional real associative division algebra is isomorphic to either the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$. In the first case we say $\mathbf{V}$ is a representation of *real type* – also called an *absolutely indecomposable* representation. In the other two cases it is called a representation of *complex* or of *quaternionic type*, respectively. We abbreviate this situation by defining the *index* of $\mathbf{V}$ to be the dimension of the division algebra from Theorem 4.6:

$$\mathrm{ind}\,(\mathbf{V}) = \dim \mathrm{End}_\Theta(\mathbf{V})/\,\mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{V}).$$

This classification can be made more precise. The fact that $\mathrm{End}_\Theta(\mathbf{V})/\,\mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{V}) \cong \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ implies that there exists $\mathbf{I} \in \mathrm{End}_\Theta(\mathbf{V})$ if $\mathbf{V}$ is of complex type and $\mathbf{I}, \mathbf{J}, \mathbf{K} \in \mathrm{End}_\Theta(\mathbf{V})$ if $\mathbf{V}$ is of quaternionic type whose equivalence classes generate the complex or quaternionic structure of the quotient algebra in (4.3). As a matter of fact, we obtain

$$\mathbf{I}^2, \mathbf{J}^2, \mathbf{K}^2, \mathbf{IJK} = -\mathbb{1}_{\mathbf{V}} + N,$$

where $N \in \mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{V})$ describes not necessarily the same nilpotent endomorphism for each left hand side. Moreover, the quotient structure implies that any $L \in \mathrm{End}_\Theta(\mathbf{V})$ is of the form

$$\begin{aligned} L &= c_1 \mathbb{1}_{\mathbf{V}} + N, \\ L &= c_1 \mathbb{1}_{\mathbf{V}} + c_I \mathbf{I} + N, \\ L &= c_1 \mathbb{1}_{\mathbf{V}} + c_I \mathbf{I} + c_J \mathbf{J} + c_K \mathbf{K} + N, \end{aligned} \tag{4.4}$$

where $N \in \mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{V})$ and $c_1, c_I, c_J, c_K \in \mathbb{R}$ depending on the representation type of $\mathbf{V}$. This characterization allows for the following simple restrictions on possible eigenvalues of endomorphisms of an arbitrary monoid representation.

Proposition 4.8. *If $\mathbf{V}$ is an absolutely indecomposable Θ-representation, any endomorphism $L \in \mathrm{End}_\Theta(\mathbf{V})$ has exactly one eigenvalue of multiplicity $\dim \mathbf{V}$ which is real. On the other hand, if $\mathbf{V}$ is a representation that has a complementable subrepresentation $W \subset \mathbf{V}$ which is indecomposable of complex type, indecomposable of quaternionic type, or the direct sum of two isomorphic subrepresentations, then there exists an endomorphism $L \in \mathrm{End}_\Theta(\mathbf{V})$ with complex eigenvalues.*

Proof. The first part follows almost directly from the well-known fact that two square matrices A, B of the same size with $\mathrm{tr}\,(A^m) = \mathrm{tr}\,(B^m)$ for all $m \in \mathbb{N}$ have the same eigenvalues with the same algebraic multiplicities using the representations of endomorphisms in (4.4). We have

$\operatorname{tr}((c_1 \mathbb{1}_{\mathbf{V}})^m) = \operatorname{tr}((c_1 \mathbb{1}_{\mathbf{V}} + N)^m)$ as $\operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{V})$ is an ideal. Hence, the only eigenvalue of all endomorphisms of this form is given by the real coefficient c_1 which is not changed by addition of the nilpotent endomorphism.

Now assume there is a complementable subrepresentation $W \subset \mathbf{V}$ of complex type, of quaternionic type, or consisting of two isomorphic absolutely indecomposable components. In each case there is an endomorphism $\mathfrak{I} \in \operatorname{End}_\Theta(W)$ such that $\mathfrak{I}^2 = -\mathbb{1}_W + N$. In the complex and quaternionic case $\mathfrak{I} = \mathbf{I}$. If $W \cong W' \oplus W'$, where W' is absolutely indecomposable, we choose

$$\mathfrak{I} = \begin{pmatrix} 0 & \mathbb{1}_{W'} \\ -\mathbb{1}_{W'} & 0 \end{pmatrix}.$$

Extending $\mathfrak{I}$ trivially to an endomorphism of $\mathbf{V}$ – i.e. $\mathfrak{I}(\overline{W}) = 0$, where $\mathbf{V} = W \oplus \overline{W}$ – yields an endomorphism $\mathfrak{I} \in \operatorname{End}_\Theta(\mathbf{V})$ which necessarily has complex eigenvalues, once again exploiting the fact stated in the beginning of this proof. $\square$

We have omitted some of the algebraic details and delicacies in these considerations (in particular of the representation (4.4)) so as to only convey the main points. To obtain the full picture we recommend to consult NIJHOLT and RINK [77].

One of the main results of this thesis describes generic steady state bifurcations in monoid equivariant dynamical systems (Theorem 5.11). We state two technical results that are essential for its proof. The upcoming proposition is an immediate consequence of the Fitting Lemma. It investigates endomorphisms of indecomposable representations that arise from concatenations of homomorphisms to and from another representation. Such an endomorphism is necessarily nilpotent, if the two representations are not isomorphic.

Proposition 4.9 (Proposition 4.2 in [91])**.** *Let $\mathbf{V}$ and $\mathbf{V}'$ be indecomposable Θ-representations, and consider two homomorphisms $L \in \operatorname{Hom}_\Theta(\mathbf{V}, \mathbf{V}')$ and $K \in \operatorname{Hom}_\Theta(\mathbf{V}', \mathbf{V})$. If $K \circ L \in \operatorname{End}_\Theta(\mathbf{V})$ is invertible, both L and K are isomorphisms. Conversely, $K \circ L \in \operatorname{End}_\Theta(\mathbf{V})$ is nilpotent if K or L is not invertible.*

Secondly, the proof of Theorem 5.11 hinges crucially on the application of the Lyapunov-Schmidt reduction as described in Section 3.4. It allows to restrict the bifurcation problem to an equivalent one on the generalized kernel of the linearization at the bifurcation point. As we have seen in Remark 4.2, the linearization at a fully synchronous steady state is an instance of an endomorphism of the representation. The final result of this section discusses perturbations of such endomorphisms and the implications on the decomposition into generalized kernel and reduced image (4.1).

Lemma 4.10 (Lemma 6.3 in [91])**.** *Let $L_0 \in \operatorname{End}_\Theta(\mathbf{V})$ and denote the decomposition into generalized kernel and reduced image of L_0 by*

$$\mathbf{V} = \ker_0(L_0) \oplus \operatorname{im}_0(L_0)$$

as in (4.1)*. Choosing a basis with respect to this representation allows to represent*

$$L_0 = \begin{pmatrix} L_0^c & 0 \\ 0 & L_0^h \end{pmatrix}$$

with L_0^c nilpotent and L_0^h invertible. Then there is an open neighborhood $U \subset \operatorname{End}_\Theta(\mathbf{V})$ of the zero endomorphism $0 \in \operatorname{End}_\Theta(\mathbf{V})$ and smooth maps

$$\phi^c \colon U \to \operatorname{End}_\Theta(\ker_0(L_0)) \quad \textit{and} \quad \phi^h \colon U \to \operatorname{End}_\Theta(\operatorname{im}_0(L_0))$$

so that every $L \in U$ represented as

$$L = \begin{pmatrix} L^c & L^{ch} \\ L^{hc} & L^h \end{pmatrix}$$

satisfies

$$L_0 + L \quad \textit{is conjugate to} \quad \begin{pmatrix} \phi^c(L) & 0 \\ 0 & \phi^h(L) \end{pmatrix} \in \mathrm{End}_\Theta(\mathbf{V}).$$

The conjugacy is realized by elements of $\mathrm{End}_\Theta(\mathbf{V})$*, which guarantees that it preserves equivariance. Furthermore, it holds that* $\phi^c(L) = L_0^c + L^c + \mathcal{O}(\|L\|^2)$ *and* $\phi^h(L) = L_0^h + L^h + \mathcal{O}(\|L\|^2)$*.*

This lemma proves that locally every endomorphism close to L_0 respects a decomposition that is equivalent to $\mathbf{V} = \ker_0(L_0) \oplus \mathrm{im}_0(L_0)$. Furthermore, up to linear order the action of L on these components depends only on its $\ker_0(L_0)$- and $\mathrm{im}_0(L_0)$-component respectively.

Remark 4.11. In Lemma 3.5 of NIJHOLT and RINK [77], the authors provide a generalization of Lemma 4.10 to a decomposition into direct sums of arbitrary generalized subspaces of L_0. In particular, this includes the decomposition into center subspace and hyperbolic subspace as in (4.2). △

4.2 Abstract representation theory for fundamental networks

We end this section with an abstractly algebraic take on the representation that equivalently characterizes the fundamental network. In particular, we consider the case of one-dimensional internal dynamics $V = \mathbb{R}$ and abbreviate $\mathbf{V} = \bigoplus_{\sigma \in \Sigma} \mathbb{R}$. Recall that the representation is given by $\sigma \to A_\sigma$ where $(A_\sigma X)_\tau = X_{\tau\sigma}$ is determined by multiplication from the right in Σ, hence the term *right regular representation*. Due to the Krull-Schmidt theorem this representation uniquely decomposes into indecomposable subrepresentations. Bifurcation analysis requires knowledge of generalized kernels or center subspaces which are subrepresentations themselves, as we have seen in Remark 4.2. These, in turn, can then uniquely be decomposed into some of the indecomposable components of $\mathbf{V}$. Hence, it is of great use for the investigation of generic bifurcations to determine the decomposition of $\mathbf{V}$. If it is known, it allows to characterize all possible generalized kernels and center subspaces in terms of symmetry. The ideas we present in this section were inspired by STEINBERG [105].

Note that the existence of a decomposition

$$\mathbf{V} = W_1 \oplus \cdots \oplus W_m$$

into indecomposable subrepresentations is equivalent to the existence of equivariant projections $P_1, \ldots, P_m \in \mathrm{End}_\Sigma(\mathbf{V})$ with $P_k(\mathbf{V}) = W_k$ that satisfy the following conditions

(i) $P_k^2 = P_k$, (projection)

(ii) $P_k P_l = 0 \in \mathrm{End}_\Sigma(\mathbf{V})$, if $k \neq l$, (direct sum)

(iii) $P_1 + \cdots + P_m = \mathbb{1}_{\mathbf{V}} \in \mathrm{End}_\Sigma(\mathbf{V})$, (decomposition)

(iv) $P_k = Q + Q'$ where Q and Q' satisfy (i) and (ii) implies $Q = 0$ or $Q' = 0$. (indecomposable)

Points (i) and (ii) guarantee that these maps are projections whose images have trivial intersection. Equivariance, furthermore, implies that the images are subrepresentations. Then, points (iii) and (iv) show that these form a decomposition of the whole space where none of the components can be decomposed any further. Hence, in order to decompose $\mathbf{V}$ it suffices to find a set of equivariant projections satisfying (i) to (iv).

In general, this is not an easier problem to solve, than to determine subrepresentations directly. Yet, we want to present some additional ideas from representation theory that give some more insight. We define what is called the *monoid algebra*

$$\mathbb{R}\Sigma = \left\{ \sum_{\sigma \in \Sigma} a_\sigma \sigma \;\middle|\; a_\sigma \in \mathbb{R} \quad \text{for all} \quad \sigma \in \Sigma \right\}$$

of formal linear combinations of the monoid elements over the real numbers. Addition is defined on coefficients per monoid element and the product structure is given by the distributive law and the multiplication in Σ. It can be seen as the real algebra with basis Σ. As the monoid is finite, it is finite-dimensional. This construction is also known in representation theory of groups. It is of great importance, since every representation can equivalently be expressed as a $\mathbb{R}\Sigma$-module. As a matter of fact, it can readily be seen that the real algebra generated by the representation maps together with addition and multiplication of endomorphisms is isomorphic to the monoid algebra

$$\langle A_\sigma \mid \sigma \in \Sigma \rangle \cong \mathbb{R}\Sigma.$$

Hence, scalar multiplication by linear combinations of representation maps gives any representation space the structure of a left $\mathbb{R}\Sigma$-module. Furthermore, the endomorphisms of a representation are precisely the endomorphisms of the corresponding module, i.e. those linear maps that respect scalar multiplication.

On the other hand, the representation space $\mathbf{V} = \bigoplus_{\sigma \in \Sigma} \mathbb{R}$ is readily seen to be isomorphic to $\mathbb{R}\Sigma$ as a vector space as well. As a matter of fact, one can identify the bases $\tau \mapsto (\delta_{\tau,\sigma})_{\sigma\in\Sigma}$, where $\mathbf{V} = \langle (\delta_{\tau,\sigma})_{\sigma\in\Sigma} \mid \tau \in \Sigma \rangle$ is the standard basis expressed using the Kronecker delta $\delta_{\tau,\sigma} = 1$ if $\tau = \sigma$ and 0 otherwise. Thus, the construction mentioned before gives rise to a left $\mathbb{R}\Sigma$-module structure on $\mathbb{R}\Sigma$. Interestingly, this is dual to the right $\mathbb{R}\Sigma$-module on $\mathbb{R}\Sigma$, where scalar multiplication is given by multiplication in $\mathbb{R}\Sigma$ from the right. It is denoted by $\mathbb{R}\Sigma_{\mathbb{R}\Sigma}$ and called the *right regular module* of $\mathbb{R}\Sigma$. Duality of modules in this context can best be understood in the sense that if we denote the scalar multiplications in terms of representation matrices using the standard basis, the matrix of the right regular *module* for each monoid element $\sigma \in \Sigma$ will be transposed to A_σ. As a matter of fact, the matrices for the right regular module respect the multiplication in Σ with reversed order so that we speak of an anti-representation.

The endomorphisms of the right regular module have a simple characterization as well. Every linear map on $\mathbb{R}\Sigma$ that respects – i.e. commutes with – multiplication from the right is given by multiplication from the left with some element in $\mathbb{R}\Sigma$. That is, $L(XY) = L(X)Y$ for all $X, Y \in \mathbb{R}\Sigma$ if and only if there exists $Z \in \mathbb{R}\Sigma$ such that $L(X) = ZX$ for all $X \in \mathbb{R}\Sigma$ (see for example Proposition A.20 in STEINBERG [105]). Hence, $\operatorname{End}(\mathbb{R}\Sigma_{\mathbb{R}\Sigma}) \cong \mathbb{R}\Sigma$ as vector spaces. In order to keep track of the order of multiplications, we introduce the *opposite algebra* $\mathbb{R}\Sigma^{\mathrm{op}}$. It contains the same elements as $\mathbb{R}\Sigma$ with reversed order of multiplication $X \bullet Y = YX$. Then we may define the right regular module $\mathbb{R}\Sigma_{\mathbb{R}\Sigma}$ as a the left $\mathbb{R}\Sigma^{\mathrm{op}}$-module ${}_{\mathbb{R}\Sigma^{\mathrm{op}}}\mathbb{R}\Sigma$ with scalar multiplication given by $\bullet$. This is due to the fact that scalar multiplication from the left reverses the order of multiplication of scalars compared to scalar multiplication from the right. Its endomorphisms then satisfy

$$\operatorname{End}\left({}_{\mathbb{R}\Sigma^{\mathrm{op}}}\mathbb{R}\Sigma\right) \cong \mathbb{R}\Sigma$$

as algebras. The right hand side is interpreted as multiplication from the left.

This can almost immediately be translated to the $\mathbb{R}\Sigma$-module structure given by the right regular representation of the fundamental network. As a matter of fact, the duality reverses the order of multiplication once more, but respects all the other relations. In particular, we obtain that the left module structure is given by $\mathbb{R}\Sigma$, i.e.

$$\langle A_\sigma \mid \sigma \in \Sigma \rangle \cong \mathbb{R}\Sigma$$

as before, but also

$$\operatorname{End}_\Sigma(\mathbf{V}) \cong \mathbb{R}\Sigma^{\mathrm{op}}.$$

Note that we have derived this second relation already by hand in Remark 4.3, where we show that the space of endomorphisms is spanned by the adjacency matrices $\{B_\sigma \mid \sigma \in \Sigma\}$. Furthermore, we see that $B_\sigma B_\tau = B_{\tau\sigma}$ so that these maps form an anti-representation. In particular $\langle B_\sigma \mid \sigma \in \Sigma \rangle \cong \mathbb{R}\Sigma^{\mathrm{op}}$.

All this brings us back to the challenge explained in the beginning of this section which is to determine a decomposition of $\mathbf{V}$ in terms of equivariant projections. As we may represent endomorphisms in terms of the opposite monoid algebra – in fact the map $\sigma \mapsto B_\sigma$ extends to an isomorphism of algebras –, we can restate this problem. Finding a decomposition of $\mathbf{V}$ into indecomposable subrepresentations is equivalent to finding a *complete set of primitive orthogonal idempotents* $e_1, \ldots, e_m \in \mathbb{R}\Sigma^{\mathrm{op}}$, where an *idempotent* is an element that squares to itself. Such a set is defined by a reformulation of the conditions (i) to (iv) as

(i) $e_k^2 = e_k$, (*idempotent*)

(ii) $e_k e_l = 0$, if $k \neq l$, (*orthogonal*)

(iii) $e_1 + \cdots + e_m = 1 (= 1 \cdot \mathrm{Id})$, (*complete*)

(iv) $e_k = q + q'$ where q and q' satisfy (i) and (ii) implies $q = 0$ or $q' = 0$. (*primitive*)

Note that these conditions are symmetric in the indices $1, \ldots, m$. Hence, reversing the order does not change any of the characteristics. Thus, the problem can furthermore be reformulated to finding a complete set of primitive orthogonal idempotents in $\mathbb{R}\Sigma$. The image under the extension of the map $\sigma \mapsto B_\sigma$ then provides a set of projections onto the indecomposable components of $\mathbf{V}$.

It was this observation that motivated a thorough investigation of the representations of fundamental feedforward networks. It turns out that these networks induce monoids with properties that allow to determine a complete set of primitive orthogonal idempotents in an algorithmic manner. The details of this can be found in Chapter 9 in Part III of this thesis. Furthermore, it has led to a purely algebraic formulation of this algorithm for so-called R-trivial monoids that is the main result of the preprint [82]. It is an open question whether other classes of networks correspond to classes of monoids for which this question can be answered in general exist.

Chapter 5

Generic monoid equivariant bifurcations

In this chapter we prove one of the main results of this thesis which is Theorem 5.11. It investigates generic steady state bifurcations in 1-parameter families of monoid equivariant dynamical systems. Recall that 1-parameter bifurcations can be seen as the standard case or the elementary building blocks of bifurcations. Even in a multi-parameter bifurcation problem generically only one parameter is responsible for a critical eigenvalue (0 or a pair of imaginary eigenvalues). Then only this parameter is relevant for the bifurcation which essentially allows to reduce the analysis to a 1-parameter bifurcation problem. The major difference occurs in the situation when multiple parameters yield critical eigenvalues simultaneously. Informally speaking, this leads to an interaction of the individual 1-parameter bifurcations, so-called *mode interactions*. This background motivates the significance of Theorem 5.11 also for multi-parameter systems. See Remark 3.30 for more on this consideration. In Section 3.4, we have seen that the Lyapunov-Schmidt reduction allows to reduce analysis of steady state bifurcations to the generalized kernel of the linearization at the bifurcation point (see Section 3.4). We also say the bifurcation occurs *along* that subspace – all branching solutions lie in the center manifold which is a graph over that subspace (see Section 3.5). Furthermore, this generalized kernel is a subrepresentation (see (4.1)). Theorem 5.11 makes this characterization more precise by proving that in a generic 1-parameter bifurcation the generalized kernel is an *absolutely indecomposable* subrepresentation. This allows for classification of 1-parameter steady state bifurcations in terms of indecomposable subrepresentations. A version for group equivariant dynamics has been known for a long time and is successfully applied for that same cause (see for example Golubitsky, Stewart, and Schaeffer [58] for an investigation of Rayleigh–Bénard convection). For monoid equivariant dynamics, however, a similar result was only known in a special case of monoid representations that decompose into pairwise non-isomorphic indecomposable subrepresentations (Rink and Sanders [91]). The generalization in Theorem 5.11 is published in [97]. Correspondingly Sections 5.1 and 5.2 are essentially Subsections 3.1 and 3.2 of that publication. Furthermore, some technicalities are postponed to the appendix and correspond to the appendix of the publication. Since then, a generalization that includes also Hopf bifurcations and bifurcations with arbitrarily many parameters has been made available. We summarize this result in Section 5.3 and also make use of it in the remainder of this thesis.

Let us briefly recall the setting and some notation first. Once again, let Θ be a monoid that acts on a finite-dimensional real vector space $\mathbf{V}$ via $\theta \mapsto T_\theta \in \mathfrak{gl}(\mathbf{V})$. Furthermore, we consider the smooth equivariant steady state bifurcation problem governed by $F\colon \mathbf{V} \times \mathbb{R} \to \mathbf{V}$. As we are only interested in the local bifurcation behavior we do not restrict the parameter space: $\Lambda = \mathbb{R}$. Symmetry is restricted to the spatial coordinate

$$F\left(T_\theta X, \lambda\right) = T_\theta F(X, \lambda)$$

for all $X \in \mathbf{V}, \lambda \in \mathbb{R}$ and $\theta \in \Theta$. We assume the existence of a Θ-symmetric equilibrium X_0 and a bifurcation to occur at the bifurcation parameter λ_0. That is the linearization $L_{\lambda_0} = D_X F(X_0, \lambda_0)$ is not invertible, i.e. $0 \in \operatorname{spec}(L_{\lambda_0})$. Without loss of generality we may assume $\lambda_0 = 0$. Then Lyapunov-Schmidt reduction tells us that possible new solution branches locally occur along the generalized kernel $\ker_0(L_0)$. Since, $L_0 \in \operatorname{End}_\Theta(\mathbf{V})$, as was mentioned in Remark 4.2, $\ker_0(L_0)$ is a subrepresentation and symmetry is preserved by this procedure so that the bifurcation problem reduces to an equivariant steady state bifurcation analysis on $\ker_0(L_0)$. Frequently, a fully symmetric steady state persists through the bifurcation. Denote it by the smooth curve $X_0(\lambda)$ such that $X_0(0) = X_0$. Then $L_\lambda = D_X F(X_0(\lambda), \lambda)$ is a one parameter family of endomorphisms of $\mathbf{V}$.

The principal part of the proof is an extension of equivariant transversality theory to generalized symmetries. To that end, we show that all endomorphisms with generalized kernel that is not absolutely indecomposable are contained in submanifolds with high codimensions. Then, a generic family of endomorphisms does not intersect these submanifolds. More precisely, any family can be perturbed so that it does not contain any elements with a generalized kernel that is not absolutely indecomposable. Note that it does not suffice to show that any endomorphism can be perturbed so that it does not have a generalized kernel of complex or of quaternionic type. This only guarantees that the a generalized kernel like that can be perturbed away at one isolated parameter value. However, it does not prevent a bifurcation along a subrepresentation of complex or of quaternionic type to occur at a different parameter value after perturbation. This proof is a generalization of the one in Rink and Sanders [91]. Therefore, its structure is similar and some of the notation is used similarly. The major difference comes from the fact that multiple indecomposable components that are isomorphic impose technical challenges in the determination of the submanifolds mentioned above.

5.1 Isotypic components and nilpotent endomorphisms

As a first step consider the decomposition of $\mathbf{V}$ into indecomposable components

$$\mathbf{V} = \mathbf{X}_1 \oplus \cdots \oplus \mathbf{X}_s$$

which is unique due to the Krull-Schmidt theorem (Theorem 4.4). Furthermore, it yields a partition of $\{1, \ldots, s\} = P_1 \cup \ldots \cup P_m$ into indices of isomorphic components, i.e $\mathbf{X}_i \cong \mathbf{X}_j$ if i and j are in the same P_k and $\mathbf{X}_i \ncong \mathbf{X}_j$ if i and j are not in the same P_k. Summing up the components according to that partition

$$V_k = \bigoplus_{i \in P_k} \mathbf{X}_i,$$

i.e. summing up those components that are isomorphic or equivalent, we obtain a coarser decomposition

$$\mathbf{V} = V_1 \oplus \cdots \oplus V_m.$$

This construction is known from representations of groups. Consequently, we use the same nomenclature by calling the V_k *isotypic components*. This decomposition is unique up to equivalence of subrepresentations as well, since it stems directly from the decomposition into indecomposable subrepresentations. As a matter of fact we may identify each isotypic component with the finite direct sum of one of its indecomposable subrepresentations

$$V_k \cong \mathbf{X}_i^{s_k}$$

with $s_k = \#P_k$ and some $i \in P_k$.

The version of Theorem 5.11 in Rink and Sanders [91] is proven for representations in which all isotypic components consist of precisely one indecomposable subrepresentation, i.e. $s_k = 1$ for all $k = 1, \ldots, m$. In order to generalize it to arbitrary isotypic components, we need to generalize the algebraic understanding of indecomposable representations such as the Fitting Lemma (Proposition 4.5), Schur's lemma (Theorem 4.6), and in particular Proposition 4.9 to isotypic components.

Proposition 5.1 (Lemma 3.1 in [97]). *Let $\mathbf{X}$ and $\mathbf{Y}$ be indecomposable Θ-representations. Furthermore, let $\mathbf{V} = \mathbf{X}^s$ and $\mathbf{W} = \mathbf{Y}^r$ for some $r, s \in \mathbb{N}$ be representations consisting of precisely one isotypic component. For $L \in \mathrm{Hom}_\Theta(V, W)$ and $K \in \mathrm{Hom}_\Theta(W, V)$ it holds that $KL \in \mathrm{End}_\Theta(\mathbf{V})$ and we may represent it as a block matrix with respect to the decomposition:*

$$KL = \begin{pmatrix} B^{11} & \cdots & B^{1s} \\ \vdots & & \vdots \\ B^{s1} & \cdots & B^{ss} \end{pmatrix}$$

so that $B^{ij} \in \mathrm{End}_\Theta(\mathbf{X})$ for all i, j. Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are non-isomorphic representations, i.e. $\mathbf{X} \ncong \mathbf{Y}$. Then all the B^{ij} are nilpotent.

Proof. As a first step, we present K and L in block matrix form respecting the decompositions of $\mathbf{V}$ and $\mathbf{W}$ respectively:

$$L = \begin{pmatrix} L^{11} & \cdots & L^{1s} \\ \vdots & & \vdots \\ L^{r1} & \cdots & L^{rs} \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} K^{11} & \cdots & K^{1r} \\ \vdots & & \vdots \\ K^{s1} & \cdots & K^{sr} \end{pmatrix}$$

where $L^{ij} \in \mathrm{Hom}_\Theta(\mathbf{X}, \mathbf{Y})$ and $K^{ij} \in \mathrm{Hom}_\Theta(\mathbf{Y}, \mathbf{X})$ for all i, j. Therefore, the product KL is an $s \times s$ block matrix with entries

$$(KL)^{ij} = \sum_{l=1}^{r} K^{il} L^{lj} \tag{5.1}$$

which are in $\mathrm{End}_\Theta(\mathbf{X})$. In particular, this holds for each summand: $K^{il}L^{lj} \in \mathrm{End}_\Theta(\mathbf{X})$ for all i, j, l. Proposition 4.5 yields that such products are either invertible or nilpotent. Suppose now that $K^{il}L^{lj}$ is invertible for some i, j, l. By Proposition 4.9 we obtain that both K^{il} and L^{lj} are isomorphisms and, therefore, $\mathbf{X} \cong \mathbf{Y}$. This is a contradiction to our assumptions and therefore $K^{il}L^{lj}$ are nilpotent for all i, j, l. The fact that $\mathrm{End}^{\mathrm{nil}}_\Theta(\mathbf{X})$ is an ideal (see Proposition 4.5) yields that finite sums of nilpotent endomorphisms are again nilpotent. In particular, this holds true for all blockwise entries of the product KL given by (5.1) which completes the proof. □

Remark 5.2. Proposition 5.1 is a generalization of Proposition 4.9 to isotypic components. △

Next we aim at understanding nilpotent endomorphisms of isotypic components similar to those of indecomposable representation (see Proposition 4.5). In order to do that we consider a real finite-dimensional representation consisting of precisely one isotypic component

$$\mathbf{V} = \mathbf{X}^s \tag{5.2}$$

(using the notation from Proposition 5.1) unless stated differently. Presenting the endomorphisms of $\mathbf{V}$ in block matrix form as before we may identify

$$\mathrm{End}_\Theta(\mathbf{V}) \cong \mathrm{M}\left(s; \mathrm{End}_\Theta(\mathbf{X})\right)$$

where $\mathrm{M}\left(s; \mathrm{End}_\Theta(\mathbf{X})\right)$ is the algebra of $s \times s$ matrices with entries in $\mathrm{End}_\Theta(\mathbf{X})$. As this identification holds for any choice of basis according to the direct sum decomposition (5.2), we usually use it without explicit mention. Then we see that the collection $\mathrm{End}^{\mathrm{b-nil}}_\Theta(\mathbf{V})$ of matrices with only nilpotent *entries*

$$\mathrm{End}^{\mathrm{b-nil}}_\Theta(\mathbf{V}) = \mathrm{M}\left(s; \mathrm{End}^{\mathrm{nil}}_\Theta(\mathbf{X})\right)$$

is an ideal in $\mathrm{End}_\Theta(\mathbf{V})$. This follows immediately from the fact that $\mathrm{End}^{\mathrm{nil}}_\Theta(\mathbf{X})$ is an ideal in $\mathrm{End}_\Theta(\mathbf{X})$ (see proposition 4.5) and from the rules of matrix summation and multiplication.

Proposition 5.3 (Lemma 3.3 in [97]). *The collection*

$$\mathrm{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V}) = \mathrm{M}\left(s; \mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right) \subset \mathrm{M}\left(s; \mathrm{End}_\Theta(\mathbf{X})\right) \cong \mathrm{End}_\Theta(\mathbf{V})$$

is an ideal.

Proof. Let

$$L = \begin{pmatrix} L^{11} & \cdots & L^{1s} \\ \vdots & & \vdots \\ L^{s1} & \cdots & L^{ss} \end{pmatrix}, K = \begin{pmatrix} K^{11} & \cdots & K^{1s} \\ \vdots & & \vdots \\ K^{s1} & \cdots & K^{ss} \end{pmatrix} \in \mathrm{M}\left(s; \mathrm{End}_\Theta(\mathbf{X})\right).$$

We may readily check the requirements for an ideal using the fact that $\mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{X})$ is an ideal.

(i) Obviously $0 \in \mathrm{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V})$.

(ii) Let $L, K \in \mathrm{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V})$. Then $L^{ij}, K^{ij} \in \mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{X})$ for all i, j. As $\mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{X})$ is an ideal, we obtain $-L^{ij}, -K^{ij} \in \mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{X})$ as well as $L^{ij} - K^{ij} \in \mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{X})$ for all i, j. Thus, $L - K \in \mathrm{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V})$.

(iii) Let $L \in \mathrm{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V})$. Then

$$(LK)^{ij} = \sum_{l=1}^{s} L^{il} K^{lj}$$

with each summand being an element in $\mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{X})$ as this is an ideal. The same holds for the finite sum and, therefore, $LK \in \mathrm{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V})$. The proof for KL is completely alike.

□

Remark 5.4. Note that the elements in $\mathrm{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V})$ are nilpotent themselves. This can be seen using the fact that after a choice of a basis for $\mathbf{X}$ an endomorphism can be represented as real matrices and

$$\mathrm{tr}\left(L^k\right) = 0$$

for all $k \in \mathbb{N}$, if $L \in \mathrm{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V})$. △

The endomorphisms in $\mathrm{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V})$ are all nilpotent. It remains to be seen, what other nilpotent endomorphisms exist in $\mathrm{End}_\Theta(\mathbf{V})$. Similar to the consideration in Schur's lemma (Theorem 4.6), we may use the fact that $\mathrm{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V})$ is an ideal in $\mathrm{End}_\Theta(\mathbf{V})$ – or more precisely in $\mathrm{M}\left(s; \mathrm{End}_\Theta(\mathbf{X})\right)$ – and consider the quotient algebra

$$\mathrm{M}\left(s; \mathrm{End}_\Theta(\mathbf{X})\right) / \mathrm{M}\left(s; \mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right).$$

It yields the decomposition

$$\mathrm{M}\left(s; \mathrm{End}_\Theta(\mathbf{X})\right) = \mathrm{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V}) \oplus \mathbf{Y}. \tag{5.3}$$

where

$$\mathbf{Y} \cong \mathrm{M}\left(s; \mathrm{End}_\Theta(\mathbf{X})\right) / \mathrm{M}\left(s; \mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right).$$

The isomorphism is the projection map

$$\pi\colon\ \mathrm{M}\left(s; \mathrm{End}_\Theta(\mathbf{X})\right) \to \mathrm{M}\left(s; \mathrm{End}_\Theta(\mathbf{X})\right) / \mathrm{M}\left(s; \mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right)$$

restricted to $\mathbf{Y}$. Note that factoring out $\mathrm{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V}) = \mathrm{M}\left(s; \mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right)$ is the same as factoring out $\mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{X})$ entrywise. Therefore,

$$\mathrm{M}\left(s; \mathrm{End}_\Theta(\mathbf{X})\right) / \mathrm{M}\left(s; \mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right) = \mathrm{M}\left(s; \mathrm{End}_\Theta(\mathbf{X}) / \mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right).$$

These algebras are not only isomorphic but equal.

Remember from the considerations after Schur's lemma (Theorem 4.6) that we may also identify

$$\operatorname{End}_\Theta(\mathbf{X}) / \operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{X}) \cong \mathbb{K}$$

where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ depending on the representation type of $\mathbf{X}$ – or equivalently on its index. Therefore, we may identify

$$\mathrm{M}\left(s; \operatorname{End}_\Theta(\mathbf{X}) / \operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right) \cong \mathrm{M}(s; \mathbb{K})$$

using an isomorphism κ. It is important to bear in mind that, even though we identify the factor space with complex or even quaternionic matrices, we still treat it as a real algebra, meaning that we allow scalar multiplication by real numbers only. This allows us to characterize all the nilpotent endomorphisms in $\operatorname{End}_\Theta(\mathbf{V})$ in terms of the reduced algebras $\mathrm{M}\left(s; \operatorname{End}_\Theta(\mathbf{X}) / \operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right) \cong \mathrm{M}(s; \mathbb{K})$.

Lemma 5.5 (Lemma 3.5 in [97]). *Let $L \in \operatorname{End}_\Theta(\mathbf{V})$. Then L is nilpotent if and only if $\pi(L)$ is nilpotent in $\mathrm{M}\left(s; \operatorname{End}_\Theta(\mathbf{X}) / \operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right)$ and $\kappa\pi(L)$ is nilpotent in $\mathrm{M}(s; \mathbb{K})$.*

Proof. The first direction of the proof follows directly from the fact that π and κ are homomorphisms of rings. Conversely let $L \in \operatorname{End}_\Theta(\mathbf{V})$ such that $\pi(L)$ is nilpotent in $\mathrm{M}\left(s; \operatorname{End}_\Theta(\mathbf{X}) / \operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right)$ (or equivalently $\kappa\pi(L)$ nilpotent in $\mathrm{M}(s; \mathbb{K})$). Then there exists $k \in \mathbb{N}$ such that

$$\pi(L)^k = 0 \in \mathrm{M}\left(s; \operatorname{End}_\Theta(\mathbf{X}) / \operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right).$$

This is the same as

$$\pi(L^k) = 0 \in \mathrm{M}\left(s; \operatorname{End}_\Theta(\mathbf{X}) / \operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right).$$

Since $\pi\colon\ \mathrm{M}(s; \operatorname{End}_\Theta(\mathbf{X})) \to \mathrm{M}(s; \operatorname{End}_\Theta(\mathbf{X})) / \mathrm{M}\left(s; \operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right)$ is the natural projection, this is equivalent to $L^k \in \operatorname{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V})$. As mentioned in the last remark, the elements of $\operatorname{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V})$ are nilpotent themselves, so L^k and therefore L are nilpotent. □

Summarizing we have seen that

$$\begin{aligned}\operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{V}) &= \{L \in \mathrm{M}(s; \operatorname{End}_\Theta(\mathbf{X})) \mid \pi(L) \text{ nilpotent}\} \\ &= \{L \in \mathrm{M}(s; \operatorname{End}_\Theta(\mathbf{X})) \mid \kappa\pi(L) \text{ nilpotent}\}.\end{aligned}$$

We can make this characterization even more precise. Recall from (5.3) that we may uniquely decompose elements $L \in \mathrm{M}(s; \operatorname{End}_\Theta(\mathbf{X}))$ as follows

$$L = L_1 + L_2, \tag{5.4}$$

where $L_1 \in \operatorname{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V})$ and $L_2 \in \mathbf{Y}$. As π is the natural projection onto the quotient algebra – in particular, it is a homomorphism –, we obtain

$$\pi(L) = \pi(L_1) + \pi(L_2) = \pi(L_2) \in \mathrm{M}\left(s; \operatorname{End}_\Theta(\mathbf{X}) / \operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right).$$

Therefore, using Lemma 5.5, we obtain

$$\begin{aligned}L = L_1 + L_2 \in \operatorname{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V}) \oplus \mathbf{Y} \text{ nilpotent} &\iff \pi(L) \in \mathrm{M}\left(s; \operatorname{End}_\Theta(\mathbf{X}) / \operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right) \text{ nilpotent} \\ &\iff \pi(L_2) \in \mathrm{M}\left(s; \operatorname{End}_\Theta(\mathbf{X}) / \operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{X})\right) \text{ nilpotent} \\ &\iff L_2 \in \mathbf{Y} \text{ nilpotent} \\ &\iff \kappa\pi(L_2) \in \mathrm{M}(s; \mathbb{K}) \text{ nilpotent}\end{aligned}$$

and in conclusion

$$\begin{aligned}\operatorname{End}^{\mathrm{nil}}_{\Theta}(\mathbf{V}) &= \{L_1 + L_2 \mid L_2 \text{ nilpotent}\} \\ &= \{L_1 + L_2 \mid \kappa\pi(L_2) \in \mathrm{M}\,(s;\mathbb{K}) \text{ nilpotent}\}\,.\end{aligned} \tag{5.5}$$

Finally, we investigate the structure of nilpotent endomorphisms of an isotypic component from a geometric perspective. The algebra of endomorphisms is first and foremost a vector space and we characterize the nilpotent ones in terms of submanifolds. In particular, their codimensions are of crucial importance. To that end, we have to investigate nilpotent matrices in $\mathrm{M}\,(s;\mathbb{K})$ due to (5.5). In the following we continue to identify endomorphisms $L \in \operatorname{End}_{\Theta}(\mathbf{V})$ with $\mathrm{M}\,(s;\operatorname{End}_{\Theta}(\mathbf{X}))$ and speak of $\pi(L)$ while remembering that this is the same as $\pi(L_2)$.

First of all, recall that matrices in $\mathrm{M}\,(s;\mathbb{K})$ are non-invertible if they are nilpotent and furthermore that they are invertible if and only if they have full rank over $\mathbb{K}$. The rank is defined to be the number of (right) linear independent column vectors or equally the number of (left) linear independent row vectors. This is well known for real and complex matrices. For the quaternionic case consult ZHANG [117] and Appendix A. This allows us to decompose $\operatorname{End}^{\mathrm{nil}}_{\Theta}(\mathbf{V})$ as follows

$$\operatorname{End}^{\mathrm{nil}}_{\Theta}(\mathbf{V}) = \bigcup_{i=1}^{s} J_i$$

where $J_i = \{L \in \operatorname{End}_{\Theta}(\mathbf{V}) \mid \kappa\pi(L) \text{ nilpotent}, \operatorname{rank}\kappa\pi(L) = s - i\}$. Furthermore, we may embed the J_i into larger collections by dropping the requirement to be nilpotent

$$J_i \subset \Lambda_i = \{L \in \operatorname{End}_{\Theta}(\mathbf{V}) \mid \operatorname{rank}\kappa\pi(L) = s - i\}\,.$$

Let $\mathrm{M}_i\,(s;\mathbb{K}) \subset \mathrm{M}\,(s;\mathbb{K})$ denote the submanifold of matrices with rank $s-i$. Its dimension and codimension are known from Proposition A.1 in Appendix A. The projections π and κ are clearly surjective linear maps and therefore submersions of manifolds, which can be seen from the decomposition in (5.3) and (5.5). Hence,

$$\Lambda_i = \pi^{-1}\kappa^{-1}(\mathrm{M}_i\,(s;\mathbb{K})) \subset \operatorname{End}_{\Theta}(\mathbf{V})$$

is a submanifold of the same codimension as that of $\mathrm{M}_i\,(s;\mathbb{K}) \subset \mathrm{M}\,(s;\mathbb{K})$:

$$\begin{aligned}\operatorname{codim}\Lambda_i &= \operatorname{codim}\mathrm{M}_i\,(s;\mathbb{K}) \\ &= i^2 \dim\mathbb{K} \\ &= i^2 \operatorname{ind}(\mathbf{X})\,.\end{aligned}$$

Recall that $i = 1,\dots,s$ and note that this codimension is 1 if and only if $i = 1$ and $\operatorname{ind}(\mathbf{X}) = 1$ or equivalently $i = 1$ and $\mathbb{K} = \mathbb{R}$. In that case we skip the embedding of J_1 into Λ_1. As is proven in Proposition A.2 in Appendix A the real nilpotent matrices of rank $s-1$ form an (s^2-s)-dimensional submanifold of $\mathrm{M}\,(s;\mathbb{R})$ that we call $\mathrm{M}^{\mathrm{nil}}_1\,(s;\mathbb{R})$. Thus, the codimension of J_1 is

$$\begin{aligned}\operatorname{codim} J_1 &= \operatorname{codim}\mathrm{M}^{\mathrm{nil}}_1\,(s;\mathbb{R}) \\ &= s^2 - (s^2 - s) \\ &= s\end{aligned}$$

using the fact that π and κ are surjective linear maps again. This codimension equals 1 if and only if $s = 1$. In this case we are considering 1×1 real matrices and the only nilpotent one is 0. We summarize these results in the following theorem.

Theorem 5.6 (Theorem 3.6 in [97]). *Let $\mathbf{X}$ be an indecomposable finite-dimensional real representation of the monoid Θ and $\mathbf{V} = \mathbf{X}^s$ for some $s \in \mathbb{N}$. Then the set of nilpotent endomorphisms $\operatorname{End}^{\mathrm{nil}}_{\Theta}(\mathbf{V})$ is contained in the finite union of submanifolds of $\operatorname{End}_{\Theta}(\mathbf{V})$ of codimensions* *(i)* s, i^2 *with* $i = 2, \dots, s$ *if* $\operatorname{ind}(\mathbf{X}) = 1$ *or*

(ii) $i^2 \operatorname{ind}(\mathbf{X})$ *with* $i = 1, \dots, s$ *if* $\operatorname{ind}(\mathbf{X}) = 2$ *or* 4.

Remark 5.7. This codimension equals 1 if and only if the representation $\mathbf{X}$ is absolutely indecomposable and the isotypic component consists of only one indecomposable summand. That is when $\operatorname{ind}(\mathbf{X}) = 1$ and $s = 1$. △

5.2 Generalized kernels in generic 1-parameter families of endomorphisms

We now return to arbitrary finite-dimensional real Θ-representations $\mathbf{V}$ decomposed into isotypic components as

$$\mathbf{V} = V_1 \oplus \dots \oplus V_m.$$

Similar to before, we may represent endomorphisms as matrices with respect to this coarser decomposition than the one into indecomposable subrepresentations, i.e. $L \in \operatorname{End}_\Theta(\mathbf{V})$ is equivalently described by

$$L = \begin{pmatrix} L^{11} & \cdots & L^{1m} \\ \vdots & & \vdots \\ L^{m1} & \cdots & L^{mm} \end{pmatrix}$$

with $L^{ij} \in \operatorname{Hom}_\Theta(V_j, V_i)$. In order to characterize the generalized kernels of endomorphisms, we need the following result on the diagonal elements of this matrix representation.

Lemma 5.8. *Let* $L \in \operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{V})$ *be represented as*

$$L = \begin{pmatrix} L^{11} & \cdots & L^{1m} \\ \vdots & & \vdots \\ L^{m1} & \cdots & L^{mm} \end{pmatrix}$$

with $L^{ij} \in \operatorname{Hom}_\Theta(V_j, V_i)$*. Then all the* $L^{ii} \in \operatorname{End}_\Theta(V_i)$ *are nilpotent –* $L^{ii} \in \operatorname{End}_\Theta^{\mathrm{nil}}(V_i)$ *for all* $i = 1, \dots, m$*.*

Proof. Let $L \in \operatorname{End}_\Theta^{\mathrm{nil}}(\mathbf{V})$ be represented as

$$L = \begin{pmatrix} L^{11} & \cdots & L^{1m} \\ \vdots & & \vdots \\ L^{m1} & \cdots & L^{mm} \end{pmatrix}$$

and $n \in \mathbb{N}$ such that $L^n = 0$. The blockwise entries of L^n are

$$(L^n)^{ij} = \sum_{1 \le l_1, \dots, l_{n-1} \le m} L^{il_1} L^{l_1 l_2} \cdots L^{l_{n-1} j}.$$

In particular, for $i = j$ we obtain

$$\begin{aligned} 0 = (L^n)^{ii} &= \sum_{1 \le l_1, \dots, l_{n-1} \le m} L^{il_1} L^{l_1 l_2} \cdots L^{l_{n-1} i} \\ &= \left(L^{ii}\right)^n + \sum_{\substack{1 \le l_1, \dots, l_{n-1} \le m \\ \exists r:\, l_r \ne i}} L^{il_1} L^{l_1 l_2} \cdots L^{l_{n-1} i}. \end{aligned}$$

Proposition 5.1 implies that

$$L^{il_1} L^{l_1 l_2} \cdots L^{l_{n-1} i} \in \operatorname{End}_\Theta^{\mathrm{b-nil}}(V_i)$$

whenever there exists an r such that $l_r \neq i$. Then all its blockwise components are nilpotent. As $\mathrm{End}_{\Theta}^{\mathrm{b-nil}}(V_i)$ is an ideal the same holds for

$$\left(L^{ii}\right)^n = - \sum_{\substack{1\leq l_1,\dots,l_{n-1}\leq m\\ \exists r\colon l_r\neq i}} L^{il_1}L^{l_1l_2}\cdots L^{l_{n-1}i}$$

In particular, this implies that $\left(L^{ii}\right)^n$ is nilpotent and thus the same holds true for L^{ii} which completes the proof. □

We have assumed that $\mathbf{V}$ splits as a sum of isotypic components. These, furthermore, decompose into indecomposable subrepresentations as follows

$$V_i \cong \mathbf{X}_i^{j_i}$$

where the $X_i \subset \mathbf{V}$ are indecomposable and $j_i \in \mathbb{N}$ suitable. If $L \in \mathrm{End}_\Theta(\mathbf{V})$ is an arbitrary endomorphism its generalized kernel $\ker_0(L)$ is a subrepresentation with a complement – the reduced image $\mathrm{im}_0(L)$. Hence, by the Krull-Schmidt theorem (Theorem 4.4) it is isomorphic to the direct sum of some of the indecomposable components of $\mathbf{V}$:

$$\ker_0(L) \cong \mathbf{X}_{i_1}^{s_1} \oplus \cdots \oplus \mathbf{X}_{i_k}^{s_k}$$

with $k \leq m$, $1 \leq i_1 < \ldots < i_k \leq m$ and suitable $1 \leq s_r \leq j_{i_r} \in \mathbb{N}$. We may, therefore, classify endomorphisms according to their generalized kernels. Renaming isotypic components of $\ker_0(L)$ as

$$W_r = \mathbf{X}_{i_r}^{s_r},$$

we denote the set of endomorphisms whose generalized kernel is isomorphic to $W_1 \oplus \cdots \oplus W_k$ by

$$\mathrm{Iso}\left(W_1 \oplus \cdots \oplus W_k\right) = \left\{L \in \mathrm{End}_\Theta(\mathbf{V}) \mid \ker_0(L) \cong W_1 \oplus \cdots \oplus W_k\right\}.$$

Using the results from Section 5.1, we may now express the set of all endomorphisms whose generalized kernel is isomorphic to a given subrepresentation geometrically in terms of submanifolds the codimensions of which can be determined from the isotypic components.

Theorem 5.9 (Theorem 3.9 in [97]). *Suppose the Θ-representation $\mathbf{V}$ decomposes as the direct sum of indecomposables*

$$V \cong \mathbf{X}_1^{j_1} \oplus \cdots \oplus \mathbf{X}_m^{j_m}$$

where the $\mathbf{X}_i$ are pairwise non-isomorphic. Choose $k \leq m$, $1 \leq i_1 < \ldots < i_k \leq m$ and $1 \leq s_r \leq j_{i_r}$ and rename $W_r = \mathbf{X}_{i_r}^{s_r}$ for $r = 1, \ldots, k$. Then $\mathrm{Iso}\left(W_1 \oplus \cdots \oplus W_k\right)$ is contained in the finite union of submanifolds of codimensions

$$\sum_{r=1}^{k} d_r \tag{5.6}$$

where

$$d_r = \begin{cases} s_r, p^2 \textit{ with } p = 2, \ldots, s_r & \textit{if} \quad \mathrm{ind}\left(\mathbf{X}_{i_r}\right) = 1 \\ p^2\,\mathrm{ind}\left(\mathbf{X}_{i_r}\right) \textit{ with } p = 1, \ldots, s_r & \textit{if} \quad \mathrm{ind}\left(\mathbf{X}_{i_r}\right) = 2 \textit{ or } 4. \end{cases}$$

Proof. Choose an arbitrary endomorphism $L_0 \in \mathrm{Iso}\left(W_1 \oplus \cdots \oplus W_k\right)$ and decompose $\mathbf{V}$ into the generalized kernel and reduced image of L_0:

$$\mathbf{V} = \ker_0(L_0) \oplus \mathrm{im}_0(L_0).$$

Recall from Lemma 4.10 that for an endomorphism $L \in \mathrm{End}_\Theta(\mathbf{V})$ close enough to the zero endomorphism $0 \in \mathrm{End}_\Theta(\mathbf{V})$ the sum $L_0 + L$ is equivariantly conjugate to

$$\begin{pmatrix} \phi^{11}(L) & 0 \\ 0 & \phi^{22}(L) \end{pmatrix}$$

with respect to that decomposition. As $\phi^{22}(L) = L_0^{22} + \mathcal{O}(\|L\|)$ and L_0^{22} invertible, $\phi^{22}(L)$ is invertible as well. The generalized kernel $\ker_0(L_0 + L)$ is therefore isomorphic to $\ker_0(L_0)$ if and only if $\phi^{11}(L)$ is nilpotent. As a matter of fact this means

$$L_0 + L \in \mathrm{Iso}\,(W_1 \oplus \cdots \oplus W_k) \iff \phi^{11}(L) \text{ nilpotent}$$

for all L in a suitable neighborhood U of $0 \in \mathrm{End}_\Theta(\mathbf{V})$.

Furthermore,

$$\phi^{11} \colon U \to \mathrm{End}_\Theta(\ker_0(L_0)) \cong \mathrm{End}_\Theta(W_1 \oplus \cdots \oplus W_k)$$

with $\phi^{11}(L) = L_0^{11} + L^{11} + \mathcal{O}(\|L\|^2)$. Hence ϕ^{11} is clearly a submersion and it suffices to prove that

$$\mathrm{End}_\Theta^{\mathrm{nil}}(W_1 \oplus \cdots \oplus W_k)$$

is contained in the union of submanifolds of $\mathrm{End}_\Theta(W_1 \oplus \cdots \oplus W_k)$ of the specified codimensions. It then follows by an argument that has already been used before that

$$\mathrm{Iso}\,(W_1 \oplus \cdots \oplus W_k) \cap U = \left(\phi^{11}\right)^{-1} \left(\mathrm{End}_\Theta^{\mathrm{nil}}(\ker_0(L_0))\right)$$

is contained in the union of submanifolds of the same codimensions.

To prove this let $L \in \mathrm{End}_\Theta(W_1 \oplus \cdots \oplus W_k)$ be arbitrary and decomposed respecting isotypic components

$$L = \begin{pmatrix} L^{11} & \cdots & L^{1k} \\ \vdots & & \vdots \\ L^{k1} & \cdots & L^{kk} \end{pmatrix}.$$

As we have seen in Lemma 5.8 the block-diagonal elements are nilpotent if L is nilpotent. Hence, we can embed

$$\mathrm{End}_\Theta^{\mathrm{nil}}(W_1 \oplus \cdots \oplus W_k) \subset \left\{ L \in \mathrm{End}_\Theta(W_1 \oplus \cdots \oplus W_k) \mid L^{rr} \in \mathrm{End}_\Theta^{\mathrm{nil}}(W_r) \text{ for all } r = 1, \ldots, k \right\}$$

which we call $\mathfrak{N}$. Theorem 5.6 tells us that each $\mathrm{End}_\Theta^{\mathrm{nil}}(W_r)$ is contained in the finite union of submanifolds of $\mathrm{End}_\Theta(W_r)$ of codimensions

$$d_r = \begin{cases} s_r, p^2 \text{ with } p = 2, \ldots, s_r & \text{if} \quad \mathrm{ind}\,(\mathbf{X}_{i_r}) = 1 \\ p^2\, \mathrm{ind}\,(\mathbf{X}_{i_r}) \text{ with } p = 1, \ldots, s_r & \text{if} \quad \mathrm{ind}\,(\mathbf{X}_{i_r}) = 2 \text{ or } 4. \end{cases} \tag{5.7}$$

Hence $\mathfrak{N}$ is contained in the finite union of submanifolds of $\mathrm{End}_\Theta(W_1 \oplus \cdots \oplus W_k)$ of codimensions

$$\sum_{r=1}^{k} d_r$$

where the d_r are chosen as in (5.7). This completes the proof. □

Remark 5.10. Note that none of the d_r vanishes. Hence, the sum of codimensions (5.6) is 0 if and only if $k = 0$. In that case the union of submanifolds from Theorem 5.9 contains all nonsingular matrices. The sum of codimensions equals 1 if and only if $k = 1$, $s_1 = 1$ and $\mathrm{ind}\,(W_1) = 1$. In that case the nilpotent endomorphisms form a real subspace and hence a proper submanifold. They are not only contained in one. In all other cases the sum of codimensions is at least 2. △

We have now collected all measures to complete the proof of the main result of this chapter.

Theorem 5.11 (Main Theorem in [97]). *Steady state bifurcations in one-parameter families of systems that are equivariant with respect to a finite-dimensional representation of a monoid generically occur along an absolutely indecomposable subrepresentation.*

Proof. Let $\mathbf{V}$ be a finite-dimensional real representation of Θ that decomposes as the direct sum of indecomposables
$$\mathbf{V} \cong \mathbf{X}_1^{j_1} \oplus \cdots \oplus \mathbf{X}_m^{j_m}.$$
Choose $k \leq m$, $1 \leq i_1 < \ldots < i_k \leq m$ and $1 \leq s_r \leq j_{i_r}$ and rename $W_r = \mathbf{X}_{i_r}^{s_r}$ for $r = 1, \ldots, k$. Theorem 5.9 and the remark thereafter tell us that $\operatorname{Iso}(W_1 \oplus \cdots \oplus W_k)$ is a submanifold of codimension 0 if and only if $k = 0$. In that case all endomorphisms in $\operatorname{Iso}(W_1 \oplus \cdots \oplus W_k)$ are invertible so that they cannot occur as the linearization at a bifurcation point. Furthermore, it is of codimension 1 if and only if $k = 1, s_1 = 1$ and $W_{i_1} = \mathbf{X}_{i_1}$ is absolutely indecomposable. In all other cases $\operatorname{Iso}(W_1 \oplus \cdots \oplus W_k)$ is contained in the finite union of submanifolds of codimension 2 or higher. In particular, this means that

$$\zeta = \{L \in \operatorname{End}_\Theta(\mathbf{V}) \mid \ker_0(L) \neq \{0\} \text{ is absolutely indecomposable}\}$$

is the finite union of submanifolds of $\operatorname{End}_\Theta(\mathbf{V})$ of codimension 1 and

$$\eta = \{L \in \operatorname{End}_\Theta(\mathbf{V}) \mid \ker_0(L) \neq \{0\} \text{ is not absolutely indecomposable}\}$$

is contained in the finite union of submanifolds of $\operatorname{End}_\Theta(\mathbf{V})$ of codimension 2 or higher. This is due to the fact that we only have finitely many possibilities of choosing $k \leq m$, $1 \leq i_1 < \ldots < i_k \leq m$ and $1 \leq s_r \leq j_{i_r}$.

Thom's transversality theorem (compare to Hirsch [63]) then shows that ζ is intersected transversely – especially in isolated points – and η is not intersected at all by a generic one parameter family of endomorphisms. Together with the considerations at the beginning of this section this completes the proof. □

5.3 Generalization

In the preprint Nijholt and Rink [77] the authors prove a more general version of Theorem 5.9. Using methods from non-commutative algebra – especially Wedderburn's structure theorem – they are able to characterize the possible generalized kernels and *center subspaces* in an l-parameter bifurcation problem for arbitrary l. Furthermore, they obtain additional precision in showing that the set $\operatorname{Iso}(W_1 \oplus \cdots \oplus W_k)$ from Theorem 5.9 and its counterpart for center subspaces are submanifolds of specific codimensions and not only subsets thereof. As we make use of this more general result in later parts of the thesis and also for completeness we restate a theorem and a remark on bifurcations from Nijholt and Rink [77]. Setting and notation are the same as in the previous sections. In particular, we consider the finite-dimensional real Θ-representation $\mathbf{V}$ that decomposes into isotypic components. We slightly adapt notation to distinguish the representation types of the indecomposable components, i.e.

$$\mathbf{V} = V_1^{\mathrm{R}} \oplus \cdots \oplus V_{m_{\mathrm{R}}}^{\mathrm{R}} \oplus V_1^{\mathrm{C}} \oplus \cdots \oplus V_{m_{\mathrm{C}}}^{\mathrm{C}} \oplus V_1^{\mathrm{H}} \oplus \cdots \oplus V_{m_{\mathrm{H}}}^{\mathrm{H}},$$

where
$$V_i^{\mathrm{R}} \cong \left(\mathbf{X}_i^{\mathrm{R}}\right)^{s_i^{\mathrm{R}}}, \quad V_i^{\mathrm{C}} \cong \left(\mathbf{X}_i^{\mathrm{C}}\right)^{s_i^{\mathrm{C}}}, \quad V_i^{\mathrm{H}} \cong \left(\mathbf{X}_i^{\mathrm{H}}\right)^{s_i^{\mathrm{H}}}$$

and $\mathbf{X}_i^{\mathrm{R}}, \mathbf{X}_i^{\mathrm{C}}$, and $\mathbf{X}_i^{\mathrm{H}}$ are indecomposable subrepresentations of real, complex, and quaternionic type respectively.

Theorem 5.12 (Theorem 3.2 in [77]). *Let $U \subset \mathbf{V}$ be a subrepresentation satisfying*

$$U \cong \bigoplus_{i=1}^{m_{\mathrm{R}}} (\mathbf{X}_i^{\mathrm{R}})^{\rho_i} \oplus \bigoplus_{i=1}^{m_{\mathrm{C}}} (\mathbf{X}_i^{\mathrm{C}})^{\gamma_i} \oplus \bigoplus_{i=1}^{m_{\mathrm{H}}} (\mathbf{X}_i^{\mathrm{H}})^{\iota_i}$$

with $0 \leq \rho_i \leq s_i^{\mathrm{R}}, 0 \leq \gamma_i \leq s_i^{\mathrm{C}}$ and $0 \leq \iota_i \leq s_i^{\mathrm{H}}$ for every i. Define the quantities

$$K_U = \sum_{i=1}^{m_{\mathrm{R}}} \rho_i + 2 \cdot \sum_{i=1}^{m_{\mathrm{C}}} \gamma_i + 4 \cdot \sum_{i=1}^{m_{\mathrm{H}}} \iota_i, \tag{5.8a}$$

$$C_U = \sum_{i=1}^{m_{\mathrm{R}}} \lceil \rho_i/2 \rceil + \sum_{i=1}^{m_{\mathrm{C}}} \gamma_i + \sum_{i=1}^{m_{\mathrm{H}}} \iota_i, \tag{5.8b}$$

where $\lceil \cdot \rceil$ denotes the nearest larger or equal integer. As before, denote the set of endomorphisms whose generalized kernel is isomorphic to U by $\mathrm{Iso}(U) \subset \mathrm{End}_\Theta(\mathbf{V})$ and, accordingly, the set of endomorphisms whose center subspace is isomorphic to U by $\mathrm{Cen}(U) \subset \mathrm{End}_\Theta(\mathbf{V})$. Then $\mathrm{Iso}(U)$ and $\mathrm{Cen}(U)$ are unions of finite sets of conjugacy invariant submanifolds of codimension K_U or higher and C_U or higher respectively.

Remark 5.13 (Remark 5 in [77]). Just as in the beginning of this chapter, an l-parameter bifurcation problem induces a smooth family of endomorphisms $L\colon \mathbb{R}^l \to \mathrm{End}_\Theta(\mathbf{V})$ so that $L(0)$ has a non-trivial generalized kernel or a non-trivial center subspace. By Thom's transversality theorem such a family generically intersects $\mathrm{Iso}(U)$ and $\mathrm{Cen}(U)$ transversely. In particular, if $l < K_U$ or $l < C_U$ the family $L(\lambda)$ does not intersect these submanifolds at all. Hence, U does not occur as a generic generalized kernel or center subspace respectively. On the other hand, when $l \geq K_U$ or $l \geq C_U$, the submanifolds can be intersected by a generic family of endomorphisms so that U occurs as a generic generalized kernel or center subspace.

In particular, for a 1-parameter bifurcation Theorem 5.11 follows as a special case. A subrepresentation U can only occur as a generic generalized kernel, if $K_U \leq 1$. This is only fulfilled, if U consists of precisely one indecomposable component of real type. On the other hand, consider 1-parameter Hopf bifurcations. This requires a pair of purely imaginary eigenvalues at the bifurcation point and shifts the focus to the center subspace. Then a subrepresentation U can only occur as a generic center subspace, if $C_U \leq 1$, which is fulfilled in one of four cases. Either it is isomorphic to precisely one indecomposable component of any type or it is the direct sum of two isomorphic components of real type. However, note that the endomorphisms of an indecomposable representation of real type have only real eigenvalues (see Proposition 4.8). Hence, U being absolutely indecomposable excludes a pair of purely imaginary eigenvalues and is therefore not possible as the center subspace of a generic Hopf bifurcation. Only the other three possibilities – indecomposable of complex or quaternionic type or direct sum of two isomorphic components of real type – remain. In particular, this implies that mode interactions – in the sense of multiple critical eigenvalues that are not complex conjugate at the bifurcation point – in generic 1-parameter bifurcations are not possible. This would require U to consist of multiple non-isomorphic components. △

Chapter 6

Implications for coupled cell systems

In Chapter 3 we have thoroughly introduced the formalism of semigroup networks and their relations to generalized symmetry. This includes several methods to investigate dynamical systems with the underlying structure of a semigroup network. Then we have explored generalized symmetries in the form of monoid representation theory and how it can be exploited for bifurcation theory in Chapters 4 and 5. In this chapter we want to return our attention to network dynamics. To that end we briefly summarize how the dynamical techniques and the structural properties of symmetry can be joined to investigate bifurcations. Section 6.1 contains a summary of the steps that are needed to classify bifurcations of steady states in a semigroup network. NIJHOLT, RINK, and SANDERS [78] propose to take a *hidden symmetry perspective* on these types of problems. We essentially follow a similar assembly in NIJHOLT, RINK, and SANDERS [79, 81]. Furthermore, we illustrate the use of Theorem 5.11 by classifying the steady state bifurcations of an 8-cell fundamental network in Section 6.2.

6.1 Bifurcations of steady states in homogeneous coupled cell systems

The hidden symmetries introduced by the fundamental network construction allow us to classify generic bifurcations of steady states in semigroup networks in terms of symmetry. This means, that we may classify the bifurcations that are inherent to a given network structure. The steps that need to be taken can be deduced from the previous chapters in a straightforward manner, the actual analysis, however, can still be cumbersome. We present the methods here informally without precise notation and refer to the corresponding parts of the thesis for details.

Assume that we are given a semigroup network with internal phase spaces and want to classify its generic l-parameter bifurcations of steady states.

Step 1. We begin by computing the fundamental network and the action of the monoid of input maps on its total phase space as the right regular representation (Definition 3.19 and Remark 3.27).

Step 2. Then, we determine all possible generalized kernels or center subspaces – depending on which kind of bifurcation we are interested in. To that end, we need to decompose the total phase space of the fundamental network into its indecomposable subrepresentations. There is no standard way of doing so. A decomposition has to be found for the right regular representation for each monoid individually. However, certain classes of monoids exist, for which this issue can be solved in an algorithmic way, as we will see in Chapter 9. Once a decomposition is found, Theorem 5.11 and Remark 5.13 allow to determine which combinations of indecomposable components can make up the generalized kernel or center subspace in a generic l-parameter bifurcation. Recall from Remark 5.13 that special care has to be taken in the investigation of Hopf bifurcations, as subrepresentations of real type only allow for real eigenvalues when they do not occur multiple times.

Step 3. For every possible generalized kernel or center subspace the center manifold reduction shows that the problem of finding generic bifurcations of the fundamental network can be reduced to finding generic equivariant bifurcations on that specific subrepresentation (Theorem 3.32). To that end, we determine all equivariant vector fields on the generalized kernel or the center subspace respectively. This can effectively done by projecting all fundamental network vector fields to the corresponding subrepresentation.

Step 4. Then, we recover the original network from the fundamental network – or at least the input networks of every cell (Section 3.1.4) – by restriction to a dynamically invariant subspace that emerges from identification of certain coordinates, i.e. a robust synchrony subspace. As the center manifold reduction respects robust synchrony, this restriction reduces the equivariant bifurcation problem on the generalized kernel or the center subspace of the fundamental network to a generic bifurcation problem for the original network. The symmetries are given by restriction to the synchrony subspace that realizes the original network as a quotient of its fundamental network.

Step 5. Finally, we determine generic l-parameter bifurcations in the class of equivariant vector fields on the generalized kernel or the center subspace of the fundamental network restricted to the synchrony subspace that realizes the original network as a quotient of the fundamental network. That means to solve the reduced bifurcation problem for steady state or periodic solutions that arises by restricting a generic equivariant vector field to the original network.

Step 6. In order to classify *all* possible bifurcations, we repeat *Steps* 3 to 5 for every possible generalized kernel or center subspace. Note that isomorphic subrepresentations of the fundamental network can be converted into each other via an equivariant change of basis. As a result any bifurcation that occurs along one of these subrepresentation equivalently also occurs along the other and vice versa, as these are fully determined by symmetry. Furthermore, this equivalence respects the restriction to the original network.

The procedure consists of multiple reduction steps of the fundamental network. A crucial part of this is the determination of possible generalized kernels and center subspaces, in particular the decomposition of the right regular representation. In general the dimension of this representation space is $n \cdot d$ where n is the number of cells in the fundamental network and d the dimension of the internal phase space. A high-dimensional representation space like this in general allows for a large number of indecomposable components and their determination is far from trivial. Two of the main results in the remainder of this thesis are concerned with further simplification of this issue. In Chapter 7 we investigate how the dimensionality of this problem may be reduced by restriction to the case of one-dimensional internal dynamics. On a different note, we see how such a decomposition can be obtained by following an easy to apply algorithm for a specific class of networks – i.e. *feedforward networks* – in Chapter 9. First, we present an example of a fundamental network in which we classify the generic steady state bifurcations. In particular, *Steps* 1 and 5 can be omitted.

6.2 An example

In this section we illustrate the step-by-step machinery to classify bifurcations from Section 6.1 with an example. We focus on the use of Theorem 5.11. Hence, we investigate the generic steady state bifurcations in an 8-cell fundamental network as given in Figure 6.1. Note that this is the fundamental network of the running example in Section 2.5. Furthermore, this network was also used as an illustrating example in Section 4 in our publication [97]. Note that the image does not include the identity Id as an arrow color. Nevertheless, we assume each cell also depends on its own state so that we have the monoid of input maps $\Sigma = \{\mathrm{Id}, \sigma_1, \ldots, \sigma_7\}$.

We assume 1-dimensional internal phase spaces $V = \mathbb{R}$ and a parameter $\lambda \in \mathbb{R}$ so that the internal dynamics is governed by the smooth response function $f\colon \mathbb{R}^8 \times \mathbb{R} \to \mathbb{R}$. We denote the state of cell σ by $X_\sigma \in \mathbb{R}$. The corresponding parameter dependent vector field is

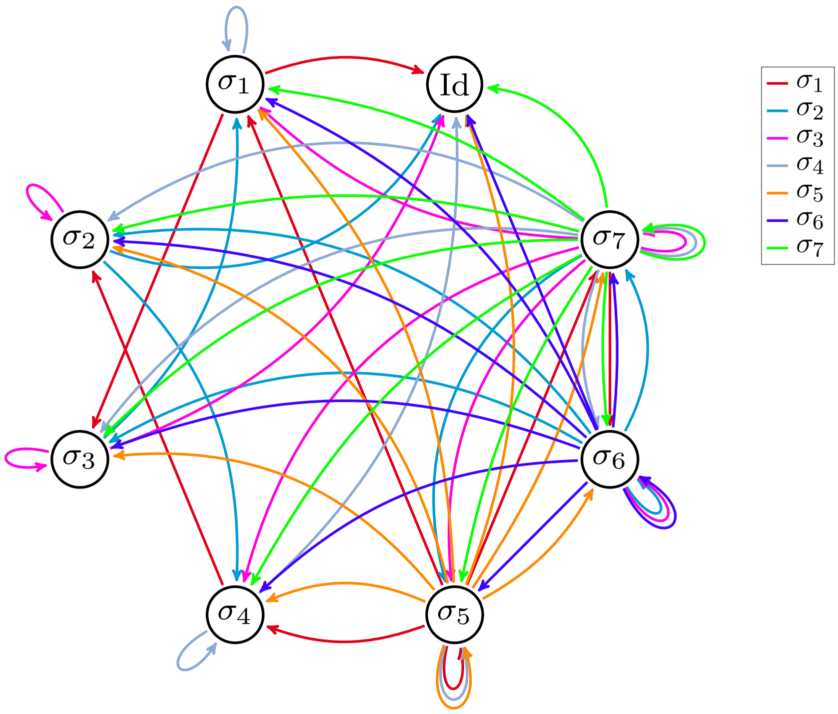


Figure 6.1: An 8-cell fundamental network.[1]

$$\Gamma_f(X,\lambda) = \begin{pmatrix} f(X_{\mathrm{Id}}, X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}, X_{\sigma_5}, X_{\sigma_6}, X_{\sigma_7}, \lambda) \\ f(X_{\sigma_1}, X_{\sigma_5}, X_{\sigma_3}, X_{\sigma_7}, X_{\sigma_1}, X_{\sigma_5}, X_{\sigma_6}, X_{\sigma_7}, \lambda) \\ f(X_{\sigma_2}, X_{\sigma_4}, X_{\sigma_6}, X_{\sigma_2}, X_{\sigma_7}, X_{\sigma_5}, X_{\sigma_6}, X_{\sigma_7}, \lambda) \\ f(X_{\sigma_3}, X_{\sigma_1}, X_{\sigma_6}, X_{\sigma_3}, X_{\sigma_7}, X_{\sigma_5}, X_{\sigma_6}, X_{\sigma_7}, \lambda) \\ f(X_{\sigma_4}, X_{\sigma_5}, X_{\sigma_2}, X_{\sigma_7}, X_{\sigma_4}, X_{\sigma_5}, X_{\sigma_6}, X_{\sigma_7}, \lambda) \\ f(X_{\sigma_5}, X_{\sigma_5}, X_{\sigma_7}, X_{\sigma_7}, X_{\sigma_5}, X_{\sigma_5}, X_{\sigma_6}, X_{\sigma_7}, \lambda) \\ f(X_{\sigma_6}, X_{\sigma_7}, X_{\sigma_6}, X_{\sigma_6}, X_{\sigma_7}, X_{\sigma_5}, X_{\sigma_6}, X_{\sigma_7}, \lambda) \\ f(X_{\sigma_7}, X_{\sigma_5}, X_{\sigma_6}, X_{\sigma_7}, X_{\sigma_7}, X_{\sigma_5}, X_{\sigma_6}, X_{\sigma_7}, \lambda) \end{pmatrix}.$$

According to Theorem 3.28, the class of parameter dependent vector fields is fully characterized by equivariance with respect to the representation $\sigma \mapsto A_\sigma$ induced by self-fibrations for every element $\sigma \in \Sigma$. That is, a parameter dependent vector field on $F\colon \mathbb{R}^8 \times \mathbb{R}^8$ is admissible if and only it is equivariant for every fixed value of λ with respect to the A_σ acting on the spatial variable

$$F_\lambda \circ A_\sigma = A_\sigma \circ F_\lambda$$

for all $\sigma \in \Sigma$ and $\lambda \in \mathbb{R}$. Furthermore, note that the monoid is generated by Id, σ_1, and σ_2

$$\Sigma = \langle \mathrm{Id}, \sigma_1, \sigma_2 \rangle.$$

Any vector field is equivariant with respect to $A_{\mathrm{Id}} = \mathbb{1}_{\mathbb{R}^8}$. Hence, in order to verify equivariance with respect to the Σ-representation it suffices to check for commutativity with respect to A_{σ_1} and A_{σ_2} – equivariance with respect to the remaining elements follows from the multiplicativity of the representation. The representation on $\mathbb{R}^8$ is therefore generated by the maps

$$\begin{aligned} A_{\sigma_1}&\colon (X_{\mathrm{Id}}, X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}, X_{\sigma_5}, X_{\sigma_6}, X_{\sigma_7}) \mapsto (X_{\sigma_1}, X_{\sigma_5}, X_{\sigma_3}, X_{\sigma_7}, X_{\sigma_1}, X_{\sigma_5}, X_{\sigma_6}, X_{\sigma_7}), \\ A_{\sigma_2}&\colon (X_{\mathrm{Id}}, X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}, X_{\sigma_5}, X_{\sigma_6}, X_{\sigma_7}) \mapsto (X_{\sigma_2}, X_{\sigma_4}, X_{\sigma_6}, X_{\sigma_2}, X_{\sigma_7}, X_{\sigma_5}, X_{\sigma_6}, X_{\sigma_7}). \end{aligned}$$

[1] Figure 6.1 is essentially Figure 4.1 in SCHWENKER [97]

To investigate steady state bifurcations we assume the existence of the trivial branch of solutions

$$F(0,\lambda)=0$$

for all parameter values $\lambda \in \mathbb{R}$. Furthermore, we want the bifurcation to occur at $\lambda_0 = 0$ which can only happen if $D_X F(0,0)$ is non-invertible due to the implicit function theorem. This means that the linearization $D_X F(0,0)$ has a non-trivial generalized kernel along which the steady state bifurcations may occur.

We want to determine all possible generalized kernels (*Step* 2). To that end, we observe that the representation of Σ decomposes into four indecomposable components

$$\mathbb{R}^8 = \mathbf{X}_1 \oplus \mathbf{X}_2 \oplus \mathbf{X}_3 \oplus \mathbf{X}_4$$

where

$$\begin{aligned}
\mathbf{X}_1 &= \{X_{\mathrm{Id}} = \ldots = X_{\sigma_7}\}\,,\\
\mathbf{X}_2 &= \{X_{\sigma_1} = \ldots = X_{\sigma_7} = 0\}\,,\\
\mathbf{X}_3 &= \{X_{\mathrm{Id}} = X_{\sigma_3}, X_{\sigma_1} = X_{\sigma_4} = X_{\sigma_5} = X_{\sigma_7} = 0\}\,,\\
\mathbf{X}_4 &= \{X_{\mathrm{Id}} = X_{\sigma_4}, X_{\sigma_2} = X_{\sigma_3} = X_{\sigma_6} = X_{\sigma_7} = 0\}\,.
\end{aligned}$$

All of these subrepresentations are of real type. This is trivial for the two one-dimensional components. Choosing the basis

$$(1,0,0,1,0,0,0,0)^T, (0,0,1,0,0,0,0,0)^T, (0,0,0,0,0,0,1,0)^T$$

for $\mathbf{X}_3$ and

$$(1,0,0,0,1,0,0,0)^T, (0,1,0,0,0,0,0,0)^T, (0,0,0,0,0,1,0,0)^T$$

for $\mathbf{X}_4$ the transformations act as matrices a_3, a_4 for σ_1 and b_3, b_4 for σ_2 on $\mathbf{X}_3$ and $\mathbf{X}_4$ respectively where

$$a_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad b_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, $\mathbf{X}_3$ and $\mathbf{X}_4$ are equivalent as subrepresentations via the isomorphism Φ with matrix representation

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

for Φ and Φ^{-1}. Furthermore, it can readily be seen that a 3×3 matrix commutes with a_3 and b_3 if and only if it is a multiple of the identity. Hence, there are no nilpotent endomorphisms and $\mathbf{X}_3$ – and due to equivalence also $\mathbf{X}_4$ – is of real type.

Theorem 5.11 tells us that generically branches of steady states bifurcate off the trivial solution along one of these components meaning that the generalized kernels of linearizations of vector fields are generically equivalent to one of the components as subrepresentations. In each case we perform the equivariant Lyapunov-Schmidt reduction or center manifold reduction to restrict to the bifurcation problem to an equivalent equivariant equation on the subrepresentation.

The subrepresentations $\mathbf{X}_1$ and $\mathbf{X}_2$ are both one-dimensional and both transformations act trivially on them. On $\mathbf{X}_1$ both σ_1 and σ_2 act as identity whereas σ_1 and σ_2 both act as zero on $\mathbf{X}_2$. Therefore, generically we observe a transcritical bifurcation in both cases. The transcritical branch along $\mathbf{X}_1$ is fully synchronous. The one on $\mathbf{X}_2$ occurs only in cell Id. This is due to the fact that cell Id has no outgoing arrows into any other cell.

As $\mathbf{X}_3$ and $\mathbf{X}_4$ are isomorphic subrepresentations, it suffices to investigate generic bifurcations along $\mathbf{X}_3$. The generic steady state bifurcations along $\mathbf{X}_4$ are the same in their specific coordinates. They only differ by their respective choice of a basis. We choose coordinates v_1, v_2, v_3 for $\mathbf{X}_3$ according to the basis above and obtain the reduced bifurcation equation

$$r(v,\lambda) = \begin{pmatrix} r_1(v_1,v_2,v_3,\lambda) \\ r_2(v_1,v_2,v_3,\lambda) \\ r_3(v_1,v_2,v_3,\lambda) \end{pmatrix} = 0$$

with the properties inherited from Γ_f

$$\begin{aligned} r(0,\lambda) = 0 \quad &\text{for all} \quad \lambda \in \mathbb{R}, \\ D_v r(0,0) \quad &\text{has an eigenvalue } 0. \end{aligned}$$

Furthermore, r is equivariant in its spatial component with respect to the monoid representation $\sigma \mapsto A_\sigma$ restricted to $\mathbf{X}_3$ – that is it is equivariant with respect to a_3 and b_3. This provides the additional properties

$$\begin{aligned} r_1(0,v_1,v_3,\lambda) &= 0, & r_1(v_2,v_3,v_3,\lambda) &= r_2(v_1,v_2,v_3,\lambda), \\ r_2(0,v_1,v_3,\lambda) &= r_1(v_1,v_2,v_3,\lambda), & r_2(v_2,v_3,v_3,\lambda) &= r_3(v_1,v_2,v_3,\lambda), \\ r_3(0,v_1,v_3,\lambda) &= r_3(v_1,v_2,v_3,\lambda), & r_3(v_2,v_3,v_3,\lambda) &= r_3(v_1,v_2,v_3,\lambda). \end{aligned}$$

These restrictions yield that up to second order we have to solve the equations

$$\begin{aligned} \alpha\lambda v_1 + \beta v_1^2 + \gamma v_1 v_3 + \mathcal{O}(\|(v,\lambda)\|^3) &= 0, \\ \alpha\lambda v_2 + \beta v_2^2 + \gamma v_2 v_3 + \mathcal{O}(\|(v,\lambda)\|^3) &= 0, \\ \alpha\lambda v_3 + (\beta+\gamma) v_3^2 + \mathcal{O}(\|(v,\lambda)\|^3) &= 0. \end{aligned} \tag{6.1}$$

Under the generic conditions that $\alpha, \beta \neq 0$ and $\gamma \neq -\beta$ this gives eight branches of solutions

$$\begin{aligned}
&v_1 = 0 && v_2 = 0 && v_3 = 0, \\
&v_1 = -\frac{\alpha}{\beta}\lambda + \mathcal{O}(\lambda^2) && v_2 = 0 && v_3 = 0, \\
&v_1 = 0 && v_2 = -\frac{\alpha}{\beta}\lambda + \mathcal{O}(\lambda^2) && v_3 = 0, \\
&v_1 = -\frac{\alpha}{\beta}\lambda + \mathcal{O}(\lambda^2) && v_2 = -\frac{\alpha}{\beta}\lambda + \mathcal{O}(\lambda^2) && v_3 = 0, \\
&v_1 = 0 && v_2 = 0 && v_3 = -\frac{\alpha}{\beta+\gamma}\lambda + \mathcal{O}(\lambda^2), \\
&v_1 = -\frac{\alpha}{\beta+\gamma}\lambda + \mathcal{O}(\lambda^2) && v_2 = 0 && v_3 = -\frac{\alpha}{\beta+\gamma}\lambda + \mathcal{O}(\lambda^2), \\
&v_1 = 0 && v_2 = -\frac{\alpha}{\beta+\gamma}\lambda + \mathcal{O}(\lambda^2) && v_3 = -\frac{\alpha}{\beta+\gamma}\lambda + \mathcal{O}(\lambda^2), \\
&v_1 = -\frac{\alpha}{\beta+\gamma}\lambda + \mathcal{O}(\lambda^2) && v_2 = -\frac{\alpha}{\beta+\gamma}\lambda + \mathcal{O}(\lambda^2) && v_3 = -\frac{\alpha}{\beta+\gamma}\lambda + \mathcal{O}(\lambda^2).
\end{aligned}$$

Returning back to the original system this implies the coexistence of eight solution branches (the trivial one and seven transcritical ones) with different cells being synchronous:

$$
\begin{aligned}
&X_{\mathrm{Id}} = X_{\sigma_1} = X_{\sigma_2} = X_{\sigma_3} = X_{\sigma_4} = X_{\sigma_5} = X_{\sigma_6} = X_{\sigma_7} = 0,\\
&X_{\mathrm{Id}} = X_{\sigma_3} = -\frac{\alpha}{\beta}\lambda + \mathcal{O}(\lambda^2), \quad X_{\sigma_1} = X_{\sigma_2} = X_{\sigma_4} = X_{\sigma_5} = X_{\sigma_6} = X_{\sigma_7} = 0,\\
&X_{\mathrm{Id}} = X_{\sigma_1} = X_{\sigma_3} = X_{\sigma_4} = X_{\sigma_5} = X_{\sigma_6} = X_{\sigma_7} = 0, \quad X_{\sigma_2} = -\frac{\alpha}{\beta}\lambda + \mathcal{O}(\lambda^2),\\
&X_{\mathrm{Id}} = X_{\sigma_2} = X_{\sigma_3} = -\frac{\alpha}{\beta}\lambda + \mathcal{O}(\lambda^2), \quad X_{\sigma_1} = X_{\sigma_4} = X_{\sigma_5} = X_{\sigma_6} = X_{\sigma_7} = 0,\\
&X_{\mathrm{Id}} = X_{\sigma_1} = X_{\sigma_2} = X_{\sigma_3} = X_{\sigma_4} = X_{\sigma_5} = X_{\sigma_7} = 0, \quad X_{\sigma_6} = -\frac{\alpha}{\beta+\gamma}\lambda + \mathcal{O}(\lambda^2),\\
&X_{\mathrm{Id}} = X_{\sigma_3} = X_{\sigma_6} = -\frac{\alpha}{\beta+\gamma}\lambda + \mathcal{O}(\lambda^2), \quad X_{\sigma_1} = X_{\sigma_2} = X_{\sigma_4} = X_{\sigma_5} = X_{\sigma_7} = 0,\\
&X_{\mathrm{Id}} = X_{\sigma_1} = X_{\sigma_3} = X_{\sigma_4} = X_{\sigma_5} = X_{\sigma_7} = 0, \quad X_{\sigma_2} = X_{\sigma_6} = -\frac{\alpha}{\beta+\gamma}\lambda + \mathcal{O}(\lambda^2),\\
&X_{\mathrm{Id}} = X_{\sigma_2} = X_{\sigma_3} = X_{\sigma_6} = -\frac{\alpha}{\beta+\gamma}\lambda + \mathcal{O}(\lambda^2), \quad X_{\sigma_1} = X_{\sigma_4} = X_{\sigma_5} = X_{\sigma_7} = 0.
\end{aligned}
$$

Note that the center manifold reduction also allows to deduce linear stability properties of the branching solutions (in the directions of the center subspace). By differentiating the reduced vector field in (6.1) we obtain the Jacobian

$$
D_v r(v_1, v_2, v_3, \lambda) = \begin{pmatrix} \mu_1 & 0 & \bullet \\ 0 & \mu_2 & \bullet \\ 0 & 0 & \mu_3 \end{pmatrix}
$$

with eigenvalues

$$
\begin{aligned}
\mu_1 &= \alpha\lambda + 2\beta v_1 + \gamma v_3 + \mathcal{O}(\|(v,\lambda)^2\|),\\
\mu_2 &= \alpha\lambda + 2\beta v_2 + \gamma v_3 + \mathcal{O}(\|(v,\lambda)^2\|),\\
\mu_3 &= \alpha\lambda + 2(\beta+\gamma) v_3 + \mathcal{O}(\|(v,\lambda)^2\|).
\end{aligned}
$$

Thus, plugging in the solutions in the same order as before, we obtain eigenvalues

$$
\begin{aligned}
&\mu_1 = +\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_2 = +\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_3 = +\alpha\lambda + \mathcal{O}(\lambda^2),\\
&\mu_1 = -\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_2 = +\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_3 = +\alpha\lambda + \mathcal{O}(\lambda^2),\\
&\mu_1 = +\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_2 = -\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_3 = +\alpha\lambda + \mathcal{O}(\lambda^2),\\
&\mu_1 = -\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_2 = -\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_3 = +\alpha\lambda + \mathcal{O}(\lambda^2),\\
&\mu_1 = +\frac{\beta}{\beta+\gamma}\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_2 = +\frac{\beta}{\beta+\gamma}\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_3 = -\alpha\lambda + \mathcal{O}(\lambda^2),\\
&\mu_1 = -\frac{\beta}{\beta+\gamma}\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_2 = +\frac{\beta}{\beta+\gamma}\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_3 = -\alpha\lambda + \mathcal{O}(\lambda^2),\\
&\mu_1 = +\frac{\beta}{\beta+\gamma}\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_2 = -\frac{\beta}{\beta+\gamma}\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_3 = -\alpha\lambda + \mathcal{O}(\lambda^2),\\
&\mu_1 = -\frac{\beta}{\beta+\gamma}\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_2 = -\frac{\beta}{\beta+\gamma}\alpha\lambda + \mathcal{O}(\lambda^2) && \mu_3 = -\alpha\lambda + \mathcal{O}(\lambda^2).
\end{aligned}
$$

Summarizing we can say that the trivial branch always switches full stability. More precisely, it exchanges stability with either branch 5 or branch 8 depending on the signs of α and $\frac{\beta}{\beta+\gamma}$. All the other branches are always saddles. The steady state bifurcations along $\mathbf{X}_4$ are the same in the respective basis.

Chapter 7

Networks with high-dimensional internal dynamics

In this final chapter of Part II we take a closer look at the influence of the choice of internal phase spaces on bifurcations of steady states and the surrounding algebraic mechanisms that we presented in the previous chapters. An issue that imposes computational difficulties in the analysis of network dynamics is the (usually) high dimension of the problem. In particular, for a semigroup network with cells $C = \{p_1, \dots, p_N\}$ and input maps $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ with internal phase space V the dimension of the total phase space is $N \cdot d$ where $d = \dim V$. Hence, commonly one restricts the investigation to the case of one-dimensional internal dynamics to reduce the dimension to its minimum while keeping the network structure intact. The results of this chapter can be summarized in the statement that the loss of generality of this restriction on the analysis of generic bifurcations of steady state is in some sense controllable. In particular, for 1-parameter steady state bifurcations the restriction to one-dimensional internal dynamics does not impose any loss of generality. To that end, we prove that the indecomposable subrepresentations of the right regular representation that is equivalent to the fundamental network are the same, independent of the dimension of the internal phase space. Each component occurs d times (Theorem 7.12). In particular, the building blocks of generalized kernels and center subspaces in generic bifurcations as in *Step* 2 in the step-by-step procedure to determine generic bifurcations described in Section 6.1 are the same in both cases. In Section 7.2 we investigate how this observation can be exploited to gain information on bifurcations in the high-dimensional case from those in the one-dimensional case. Sections 7.1 and 7.2 are essentially Sections 2 and 3 in our preprint [83].

Throughout this chapter we frequently compare properties of dynamical systems with the same underlying network structure with one- and d-dimensional internal dynamics, where $d > 1$. To avoid confusion when setting them side by side we introduce some notational conventions. In general, we distinguish these two settings by referring to them as the cases **1D** and **DD** respectively. We denote the internal phase spaces by $V = \mathbb{R}$ or $V = W \cong \mathbb{R}^d$. If we do not want to specify one of the two cases – i.e. if an observation holds for both cases simultaneously – we keep on denoting the internal phase space by V. In particular, we have coordinates in the total phase spaces given by $(v_p)_{p \in C} \in \bigoplus_{p \in C} \mathbb{R}, (w_p)_{p \in C} \in \bigoplus_{p \in C} W$ and $(x_p)_{p \in C} \in \bigoplus_{p \in C} V$. The majority of the investigations in this chapter focuses on fundamental networks. For these we drop the convention of denoting coordinates with capital letters as we did in Section 3.1.4. However, we abbreviate the total phase spaces by $\mathbf{V}^1 = \bigoplus_{\sigma \in \Sigma} \mathbb{R}$ and $\mathbf{V}^{\mathsf{D}} = \bigoplus_{\sigma \in \Sigma} W$. Consequently the coordinates on these spaces are given by $v = (v_\sigma)_{\sigma \in \Sigma} \in \mathbf{V}^1, \omega = (w_\sigma)_{\sigma \in \Sigma} \in \mathbf{V}^{\mathsf{D}}$ and $x = (x_\sigma)_{\sigma \in \Sigma} \in \bigoplus_{\sigma \in \Sigma} V$. Finally, recall that the total phase space of the fundamental network is the representation space of the right regular representation of the monoid of input maps Σ. We denote the linear maps by which its elements act on $\mathbf{V}^1$ and $\mathbf{V}^{\mathsf{D}}$ by A^1_σ and A^{D}_σ respectively. They are defined by

$$(A^1_\sigma v)_\tau = v_{\tau\sigma}, \qquad (A^{\mathsf{D}}_\sigma \omega)_\tau = w_{\tau\sigma}.$$

7.1 Dimension reduction in fundamental networks

We begin by investigating the relation of the right regular representations of Σ in the cases **1D** and **DD**. In particular, we show that the indecomposable components are the same in both cases which leads to 'the same' generalized kernels and center subspaces in bifurcation analysis. One of the main tools in this section is a representation of the total phase space in the case **DD** in tensor product notation. To that end, we may identify

$$\bigoplus_{p\in C} W \cong \left(\bigoplus_{p\in C} \mathbb{R}\right) \otimes W. \tag{7.1}$$

This can best be seen using the standard basis

$$\bigoplus_{p\in C} \mathbb{R} = \langle (\delta_{q,p})_{p\in C} \mid q \in C\rangle$$

encoded in the Kronecker delta as before. Then we may split each $(w_p)_{p\in C} \in \bigoplus_{i=1}^N W$ as a sum of vectors that have precisely one non-vanishing coordinate entry: $(w_p)_{p\in C} = \sum_{q\in C}(\delta_{q,p}w_q)_{p\in C}$. This can be represented as a sum of pure tensors

$$\sum_{q\in C}(\delta_{q,p})_{p\in C} \otimes w_q. \tag{7.2}$$

Note that this *notation* as the sum of pure tensors is not unique, since multiple sums of pure tensors can represent the same element $(w_p)_{p\in C} \in \bigoplus_{p\in C} W$. For example

$$(\delta_{p,q})_{p\in C} \otimes sw_p = s(\delta_{p,q})_{p\in C} \otimes w_p$$

for a scalar $s \in \mathbb{R}$. However, in critical cases like this the tensors are identified via an equivalence relation. Hence, the *representation* via tensors in (7.2) is unique. For more on this see the details and definitions of tensor products of vector spaces (e.g. in JACOBSON [64]). This formalism allows for a graphical interpretation of attaching a vector space to each cell instead of using vectors of vectors and block matrices. It has been used in AGUIAR and RUAN [11], DIAS and LAMB [27], GOLUBITSKY and LAUTERBACH [42], and LEITE and GOLUBITSKY [71] to describe adjacency matrices and their spectral properties for networks with high-dimensional internal dynamics. Note that the isomorphism in (7.1) is only an identification of different notations.

Remark 7.1. The representation (7.2) relies on the fact that for two finite-dimensional vector spaces $\mathbf{V}$ and $\mathbf{V}'$ with bases $\{b_i\}_{i\in I}$ and $\{b'_j\}_{j\in J}$ the elements $\{b_i \otimes b'_j\}_{i\in I, j\in J}$ form a basis for the tensor product $\mathbf{V} \otimes \mathbf{V}'$. In particular, given a basis $\{b_1, \dots, b_d\}$ for W, we can represent each element $(w_p)_{p\in C} \in \bigoplus_{p\in C} W$ as

$$\sum_{p\in C}(\delta_{p,q})_{q\in C} \otimes w_p = \sum_{p\in C}\sum_{i=1}^{d}(\delta_{p,q})_{q\in C} \otimes \alpha_i^p \cdot b_i = \sum_{p\in C}\sum_{i=1}^{d} \alpha_i^p \cdot (\delta_{p,q})_{q\in C} \otimes b_i,$$

where $w_p = \alpha_1^p b_1 + \cdots + \alpha_d^p b_d$. △

The tensor product formalism allows to give a precise characterization of linear admissible maps in the case **DD**. Recall from Proposition 3.8 that any linear admissible map $L\colon \bigoplus_{p\in C} V \to \bigoplus_{p\in C} V$ can be represented using the generalized adjacency matrices B_σ defined by $(B_\sigma(x_p)_{p\in C})_q = x_{\sigma(q)}$. Using the tensor notation, we see that these matrices from the case **1D** are sufficient to represent linear admissible maps also in the case **DD**.

Proposition 7.2. *Define the* **1D** *linear map* $B^1_\sigma\colon \bigoplus_{p\in C}\mathbb{R} \to \bigoplus_{p\in C}\mathbb{R}$ *via* $(B^1_\sigma(v_p)_{p\in C})_q = v_{\sigma(q)}$ *for all* $\sigma\in\Sigma$*. Then any linear admissible map in the case* **DD** $L\colon \bigoplus_{p\in C} W \to \bigoplus_{p\in C} W$ *is of the form*

$$L = \sum_{\sigma\in\Sigma} B^1_\sigma \otimes b_\sigma \tag{7.3}$$

for suitable linear maps $b_\sigma \in \mathfrak{gl}(W)$.

Proof. Let $L\colon \bigoplus_{p\in C} W \to \bigoplus_{p\in C} W$ be a linear admissible map. According to Proposition 3.8 there are linear maps $b_\sigma \in \mathfrak{gl}(W)$ for each $\sigma\in\Sigma$ such that

$$(L((w_q)_{q\in C}))_p = \sum_{\sigma\in\Sigma} b_\sigma(B^{\mathsf{D}}_\sigma(w_q)_{q\in C})_p = \sum_{\sigma\in\Sigma} b_\sigma(w_{\sigma(p)})$$

in the non-tensor notation, where $(B^{\mathsf{D}}_\sigma(w_q)_{q\in C})_p = w_{\sigma(p)}$. In the tensor notation (7.2), $L((w_q)_{q\in C})$ can therefore be represented as

$$L((w_q)_{q\in C}) = \sum_{p\in C}(\delta_{p,q})_{q\in C} \otimes \sum_{\sigma\in\Sigma} b_\sigma(w_{\sigma(p)}) = \sum_{p\in C}\sum_{\sigma\in\Sigma}(\delta_{p,q})_{q\in C}\otimes b_\sigma(w_{\sigma(p)}). \tag{7.4}$$

On the other hand $\left(B^1_\sigma(\delta_{p,q})_{q\in C}\right)_r = \delta_{p,\sigma(r)}$, which equals 1 if $r\in\sigma^{-1}(p)$ and 0 otherwise. In particular,

$$B^1_\sigma(\delta_{p,q})_{q\in C} = \sum_{r\in\sigma^{-1}(p)} (\delta_{r,q})_{q\in C}.$$

Using the tensor notation (7.2), i.e. representing $(w_q)_{q\in C}$ as

$$\sum_{p\in C}(\delta_{p,q})_{q\in C}\otimes w_p,$$

we compute

$$\begin{aligned}
\left[\sum_{\sigma\in\Sigma} B^1_\sigma\otimes b_\sigma\right]\left(\sum_{p\in C}(\delta_{p,q})_{q\in C}\otimes w_p\right) &= \sum_{\sigma\in\Sigma}\sum_{p\in C} B^1_\sigma(\delta_{p,q})_{q\in C}\otimes b_\sigma(w_p)\\
&= \sum_{\sigma\in\Sigma}\sum_{p\in C}\sum_{r\in\sigma^{-1}(p)}(\delta_{r,q})_{q\in C}\otimes\left(b_\sigma w_{\sigma(r)}\right)\\
&= \sum_{\sigma\in\Sigma}\sum_{r\in C}(\delta_{r,q})_{q\in C}\otimes\left(b_\sigma w_{\sigma(r)}\right).
\end{aligned} \tag{7.5}$$

Therein the last equation holds since $\left\{\sigma^{-1}(p) \mid p\in C\right\}$ forms a partition of C for all $\sigma\in\Sigma$. As (7.4) and (7.5) agree, this completes the proof. □

Remark 7.3. Note that the tensor notation is also applicable in the case **1D**. Then in (7.2) and (7.3) we tensor with a scalar. Furthermore, $\mathfrak{gl}(\mathbb{R})\cong\mathbb{R}$ so that application of a linear map can be identified with scalar multiplication. This only plays a role in cases where we do not explicitly distinguish between **1D** and **DD**. △

Furthermore, we may also describe synchrony subspaces in the tensor notation. We use the characterization of robust synchrony patterns in terms of balanced partitions from Proposition 3.11.

Proposition 7.4. *Let* $P = \{P_1,\dots,P_r\}$ *be a balanced partition of the cells* C *and let*

$$\Delta^1_P = \left\{ (v_p)_{p\in C} \in \bigoplus_{p\in C}\mathbb{R} \;\middle|\; v_p = v_q, \quad \textit{if} \quad p,q\in P_i \quad \textit{for some } 1\le i\le r\right\},$$

$$\Delta^{\mathsf{D}}_P = \left\{ (w_p)_{p\in C} \in \bigoplus_{p\in C} W \;\middle|\; w_p = w_q, \quad \textit{if} \quad p,q\in P_i \quad \textit{for some } 1\le i\le r\right\}$$

be the corresponding synchrony subspaces for one- and for high-dimensional internal dynamics respectively. Then

$$\Delta_P^{\mathsf{D}} \cong \Delta_P^{\mathbb{1}} \otimes W.$$

Proof. The result follows almost directly from the characterization of bases of tensor products in Remark 7.1. The synchrony subspace $\Delta_P^{\mathbb{1}}$ is spanned by elements $\{(v_p^1)_{p\in C}, \ldots, (v_p^r)_{p\in C}\}$, where $v_p^i = 1$ if $p \in P_i$ and $v_p^i = 0$ otherwise. Hence,

$$\left\{ (v_p^i)_{p\in C} \otimes b_j \;\middle|\; i = 1, \ldots, r \text{ and } j = 1, \ldots, d \right\} \tag{7.6}$$

is a basis of $\Delta_P^{\mathbb{1}} \otimes W$, when $\{b_1, \ldots, b_d\}$ is a basis of W.

On the other hand, let $p_1, \ldots, p_r \in C$ be a set of representatives of the partition P, i.e. $p_i \in P_i$. Then every element $(w_p)_{p\in C} \in \Delta_P^{\mathsf{D}}$ can be represented as

$$(w_p)_{p\in C} = \sum_{i=1}^{r} (v_p^i \cdot w_{p_i})_{p\in C}$$

using the basis of $\Delta_P^{\mathbb{1}}$, since $w_p = w_{p_i}$ if $p \in P_i$. Furthermore, every element w_{p_i} is of the form

$$w_{p_i} = \sum_{j=1}^{d} \alpha_j^i \cdot b_j$$

using the basis of W. Hence, we obtain

$$(w_p)_{p\in C} = \sum_{i=1}^{r} \sum_{j=1}^{d} \alpha_j^i \cdot (v_p^i \cdot b_j)_{p\in C}.$$

In particular, we see that

$$\left\{ (v_p^i \cdot b_j)_{p\in C} \;\middle|\; i = 1, \ldots, r \text{ and } j = 1, \ldots, d \right\} \tag{7.7}$$

is a basis of Δ_P^{D}. Representing these basis elements in their respective tensor notation (7.2) shows that (7.7) agrees with (7.6) which completes the proof. $\square$

From now on, we focus on fundamental networks. We want to understand the structure of the regular representation and its decomposition into indecomposable subrepresentations

$$\bigoplus_{\sigma\in\Sigma} V = W_1 \oplus \cdots \oplus W_m. \tag{7.8}$$

As it turns out, this decomposition in the case **DD** is strongly related to that in the case **1D**. Our choice of bases for the total phase spaces reflects the network structure. As in Remark 3.27 the representation maps $A_\sigma^{\mathbb{1}}, A_\sigma^{\mathsf{D}}$ can be interpreted as $n \times n$-matrices with entries in $\mathbb{R}$ or $\mathfrak{gl}(W)$ respectively in these bases. Obviously $A_\sigma^{\mathbb{1}}$ and A_σ^{D} have a very similar structure. The matrices $A_\sigma^{\mathbb{1}}$ have an entry 1 per row and all the other entries equal to 0. The matrices A_σ^{D} can be interpreted as matrices with the same structure with entries in $\mathfrak{gl}(W)$. Hence, whenever $A_\sigma^{\mathbb{1}}$ has an entry 1, A_σ^{D} has an entry $\mathbb{1}_W$. Accordingly an entry 0 in $A_\sigma^{\mathbb{1}}$ corresponds to an entry $0 \in \mathfrak{gl}(W)$ of A_σ^{D}. The following proposition further explores the relation between these two representations using the tensor formalism (7.1).

Proposition 7.5. *The Σ-representation $\{A_\sigma^{\mathsf{D}}\}_{\sigma\in\Sigma}$ on $\mathbf{V}^{\mathsf{D}}$ is isomorphic to $\mathbf{V}^{\mathbb{1}} \otimes W$ on which $\sigma \in \Sigma$ acts as $A_\sigma^{\mathbb{1}} \otimes \mathbb{1}_W$.*

Proof. The main idea for the proof is the interpretation of the total phase space $\mathbf{V}^{\mathsf{D}}$ as having the vector space W attached to each cell of the network. These are in 1-to-1 correspondence to the coordinates in $(v_\sigma)_{\sigma\in\Sigma} \in \mathbf{V}^1 = \bigoplus_{\sigma\in\Sigma}\mathbb{R}$. Hence, we assign a vector $w_\sigma \in W$ to each coordinate v_σ which is reflected in the tensor notation $\mathbf{V}^1 \otimes W$.

First, note that both vector spaces have dimension $n\cdot\dim W$. Hence, they are isomorphic as such. An isomorphism can be defined as

$$\begin{aligned}\Phi\colon \mathbf{V}^1 \otimes W &\to \mathbf{V}^{\mathsf{D}}\\ (v_\sigma)_{\sigma\in\Sigma} \otimes w &\mapsto (v_\sigma w)_{\sigma\in\Sigma},\end{aligned}$$

which is linearly extended to non-pure tensors (sums of elements $(v_\sigma)_{\sigma\in\Sigma} \otimes w$). Note that this isomorphism coincides with the often used identification of $(v_\sigma)_{\sigma\in\Sigma} \otimes w$ with the outer product $(v_\sigma)_{\sigma\in\Sigma} w^T = (v_\sigma w^T)_{\sigma\in\Sigma}$, except for the transposition in the W-component.

As in (7.2), we may uniquely split each $\omega = (w_\sigma)_{\sigma\in\Sigma} \in \mathbf{V}^{\mathsf{D}}$ as a sum of vectors that have precisely one non-vanishing coordinate entry: $(w_\sigma)_{\sigma\in\Sigma} = \sum_{\tau\in\Sigma}(\delta_{\sigma,\tau}w_\sigma)_{\sigma\in\Sigma}$. Hence, the map

$$\Psi\colon (w_\sigma)_{\sigma\in\Sigma} \mapsto \sum_{\tau\in\Sigma}(\delta_{\sigma,\tau})_{\sigma\in\Sigma} \otimes w_\tau$$

is inverse to Φ. In particular,

$$\begin{aligned}\Psi\left(\Phi((v_\sigma)_{\sigma\in\Sigma} \otimes w)\right) &= \Psi\left((v_\sigma w)_{\sigma\in\Sigma}\right)\\ &= \Psi\left(\sum_{\tau\in\Sigma}(\delta_{\sigma,\tau}v_\sigma w)_{\sigma\in\Sigma}\right)\\ &= \sum_{\tau\in\Sigma}(\delta_{\sigma,\tau})_{\sigma\in\Sigma} \otimes (v_\sigma w)\\ &= \sum_{\tau\in\Sigma}(\delta_{\sigma,\tau}v_\sigma)_{\sigma\in\Sigma} \otimes w\\ &= (v_\sigma)_{\sigma\in\Sigma} \otimes w,\end{aligned}$$

which extends linearly to non-pure tensors. Recall that the basis $\{(\delta_{\sigma,\tau})_{\sigma\in\Sigma}\}_{\tau\in\Sigma} \subset \mathbf{V}^1$ corresponds to the cells of the network. Therefore, we may interpret Ψ as the map that picks the vector w_σ in the σ-entry and attaches it to cell σ via the tensor product.

It remains to be checked that Φ intertwines the two Σ-representations. In order to do so, we compute

$$\Phi\left([A^1_\tau \otimes \mathbb{1}_W]((v_\sigma)_{\sigma\in\Sigma} \otimes w)\right) = \Phi((v_{\sigma\tau})_{\sigma\in\Sigma} \otimes w) = (v_{\sigma\tau}w)_{\sigma\in\Sigma}$$

but also

$$A^{\mathsf{D}}_\tau\Phi((v_\sigma)_{\sigma\in\Sigma} \otimes w) = A^{\mathsf{D}}_\tau(v_\sigma w)_{\sigma\in\Sigma} = (v_{\sigma\tau}w)_{\sigma\in\Sigma}.$$

Equivariance on non-pure tensors follows from linearity of the representation matrices. This proves equivalence of the representations. □

Remark 7.6. Propositions 7.2 and 7.5 allow for a characterization of endomorphisms of the right regular representation that is equivalent to the fundamental network structure as in Corollary 3.23 and Remark 4.3. For those networks the adjacency matrices are defined by multiplication from the left in Σ as $(B^1_\sigma(v))_\tau = v_{\sigma\tau}$. As the linear admissible maps are precisely those linear maps that commute with the representation, we know in the tensor notation any endomorphism $L \in \operatorname{End}_\Sigma\left(\mathbf{V}^1 \otimes W\right)$ is of the form (7.3), i.e.

$$L = \sum_{\sigma\in\Sigma} B^1_\sigma \otimes b_\sigma$$

for linear maps $b_\sigma \in \mathfrak{gl}(W)$. △

The tensor notation relates the representation $\mathbf{V}^{\mathsf{D}}$ corresponding to the fundamental network with d-dimensional internal dynamics in a straightforward way to $\mathbf{V}^1$ in the case $\mathbf{1D}$. Understanding the structure of the representation, especially its decomposition into subrepresentations, is essential for the investigation of generic dynamics. The following results relate the decomposition of $\mathbf{V}^1 \otimes W$ to that of $\mathbf{V}^1$. The first lemma serves as a reminder of a well-known result on the decomposition of tensor products of vector spaces.

Lemma 7.7. *Let $\mathbf{V}$ and $\mathbf{V}'$ be vector spaces with $\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2$. Then $\mathbf{V} \otimes \mathbf{V}' = (\mathbf{V}_1 \otimes \mathbf{V}') \oplus (\mathbf{V}_2 \otimes \mathbf{V}')$.*

Proof. The decomposition $\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2$ is equivalent to the existence of projection operators $\pi_1, \pi_2 \in \mathfrak{gl}(\mathbf{V})$ such that

$$\begin{aligned}
&\mathrm{im}\,(\pi_1) = \mathbf{V}_1, \\
&\mathrm{im}\,(\pi_2) = \mathbf{V}_2, \\
&\pi_1 + \pi_2 = \mathbb{1}_{\mathbf{V}} \quad \text{and} \\
&\pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 = 0 \in \mathfrak{gl}(\mathbf{V}).
\end{aligned}$$

It is easy to check that $\pi_1 \otimes \mathbb{1}_{\mathbf{V}'}, \pi_2 \otimes \mathbb{1}_{\mathbf{V}'} \in \mathfrak{gl}(\mathbf{V} \otimes \mathbf{V}')$ are projections as well with

$$\begin{aligned}
\mathrm{im}\,(\pi_1 \otimes \mathbb{1}_{\mathbf{V}'}) &= \mathbf{V}_1 \otimes \mathbf{V}', \\
\mathrm{im}\,(\pi_2 \otimes \mathbb{1}_{\mathbf{V}'}) &= \mathbf{V}_2 \otimes \mathbf{V}'.
\end{aligned}$$

Furthermore, we have

$$\pi_1 \otimes \mathbb{1}_{\mathbf{V}'} + \pi_2 \otimes \mathbb{1}_{\mathbf{V}'} = (\pi_1 + \pi_2) \otimes \mathbb{1}_{\mathbf{V}'} = \mathbb{1}_{\mathbf{V}} \otimes \mathbb{1}_{\mathbf{V}'}$$

and

$$\begin{aligned}
(\pi_1 \otimes \mathbb{1}_{\mathbf{V}'}) \circ (\pi_2 \otimes \mathbb{1}_{\mathbf{V}'}) &= ((\pi_1 \circ \pi_2) \otimes \mathbb{1}_{\mathbf{V}'}) = 0 \otimes \mathbb{1}_{\mathbf{V}'} = 0, \\
(\pi_2 \otimes \mathbb{1}_{\mathbf{V}'}) \circ (\pi_1 \otimes \mathbb{1}_{\mathbf{V}'}) &= ((\pi_2 \circ \pi_1) \otimes \mathbb{1}_{\mathbf{V}'}) = 0 \otimes \mathbb{1}_{\mathbf{V}'} = 0.
\end{aligned}$$

Hence, $\mathbf{V} \otimes \mathbf{V}' = (\mathbf{V}_1 \otimes \mathbf{V}') \oplus (\mathbf{V}_2 \otimes \mathbf{V}')$. □

Remark 7.8. The same argumentation proves $\mathbf{V}' = \mathbf{V}'_1 \oplus \mathbf{V}'_2 \implies \mathbf{V} \otimes \mathbf{V}' = (\mathbf{V} \otimes \mathbf{V}'_1) \oplus (\mathbf{V} \otimes \mathbf{V}'_2)$. △

Remark 7.9. If, in the above setting, $\mathbf{V}^1 = Y_1 \oplus Y_2$ where Y_1 and Y_2 are subrepresentations, the projections π_1 and π_2 in the previous proof are equivariant with respect to $\{A^1_\sigma\}_{\sigma \in \Sigma}$. Then $\pi_1 \otimes \mathbb{1}_W$ and $\pi_2 \otimes \mathbb{1}_W$ are equivariant with respect to $\{A^1_\sigma \otimes \mathbb{1}_W\}_{\sigma \in \Sigma}$. Thus, $\mathbf{V}^1 \otimes W = (Y_1 \otimes W) \oplus (Y_2 \otimes W)$ as a decomposition into subrepresentations. The same constructions works for a decomposition of W. △

Lemma 7.10. *Suppose $Y \subset \mathbf{V}^1$ is a subrepresentation with respect to $\{A^1_\sigma\}_{\sigma \in \Sigma}$ and let $w \in W \setminus \{0\}$ be an arbitrary element. Then $Y \otimes \langle w \rangle \subset \mathbf{V}^1 \otimes V$ is a subrepresentation with respect to $\{A^1_\sigma \otimes \mathbb{1}_W\}_{\sigma \in \Sigma}$. Furthermore, $Y \otimes \langle w \rangle \cong Y$ as subrepresentations.*

Proof. We first note

$$\begin{aligned}
y_1 \otimes r_1 w + \cdots + y_k \otimes r_k w &= r_1 y_1 \otimes w + \cdots + r_k y_k \otimes w \\
&= (r_1 y_1 + \cdots + r_k y_k) \otimes w
\end{aligned}$$

for an arbitrary formal sum in $Y \otimes \langle w \rangle$, i.e. $y_1, \ldots, y_k \in Y$ and $r_1, \ldots, r_k \in \mathbb{R}$. Hence, every element in $Y \otimes \langle w \rangle$ can uniquely be expressed as a pure tensor $y \otimes w$ with $y = (r_1 y_1 + \cdots + r_k y_k) \in Y$. Therefore, we may identify $y \otimes w$ with y which is equivariant by definition of the representation maps. This proves the claim. □

Remark 7.11. In particular, if Y is indecomposable of a specific type, the same holds for $Y \otimes \langle w \rangle$, due to equivalence of the representations. $\triangle$

We obtain the main result of this section as a corollary of the above.

Theorem 7.12. *Suppose $\mathbf{V}^1 = Y_1 \oplus \cdots \oplus Y_s$ is a decomposition into indecomposable subrepresentations with respect to $\{A^1_\sigma\}_{\sigma\in\Sigma}$. Let $\{b_1, \ldots, b_d\}$ be a basis for W. Then*

$$\mathbf{V}^{\mathsf{D}} \cong \mathbf{V}^1 \otimes W = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{d} Y_i \otimes \langle b_j \rangle \tag{7.9a}$$

$$\cong \bigoplus_{i=1}^{s} Y_i^d \tag{7.9b}$$

as a decomposition into indecomposable subrepresentations with respect to $\{A_\sigma \otimes \mathbb{1}_W\}_{\sigma\in\Sigma}$.

Remark 7.13. Note that the first isomorphism in (7.9a) is only an identification of different notations. The second relation however is an equality. Hence, we may identify the fundamental network representation space $\mathbf{V}^{\mathsf{D}}$ in the case **DD** with the decomposition given in that equation without changing the coordinates. In particular the network structure is preserved by this identification. On the other hand, not every decomposition of $\mathbf{V}^{\mathsf{D}}$ is of the form (7.9a). By the Krull-Schmidt theorem (Theorem 4.4) every indecomposable component $W_i \subset \mathbf{V}^{\mathsf{D}}$ is isomorphic to one of $\mathbf{V}^1$, i.e. $W_i \cong Y_i \otimes \langle w \rangle \cong Y_i$ after relabeling the indices, but they are not necessarily equal. Nonetheless, every decomposition of $\mathbf{V}^1$ gives rise to a decomposition of $\mathbf{V}^{\mathsf{D}}$ as in Theorem 7.12. $\triangle$

Corollary 7.14. *Let W^1 and W^2 be two finite-dimensional real vector spaces that we choose as internal phase spaces of a fundamental network. Furthermore, assume $\dim W^1 \leq \dim W^2$. Then there is a subrepresentation $U \subset \bigoplus_{\sigma\in\Sigma} W^2$ such that*

$$U \cong \bigoplus_{\sigma\in\Sigma} W^1.$$

Proof. This follows directly from (7.9b). $\square$

Finally, we show that the identification of indecomposable subrepresentations in the cases **1D** and **DD** respects synchrony subspaces. This is particularly relevant in the analysis of bifurcations, as it implies that the restriction to the case **1D** does not change patterns of synchrony. More precisely, recall from Remark 3.33 that the center manifold reduction respects synchrony subspaces. Hence, restriction to one of these synchrony subspaces yields a reduced bifurcation problem within the corresponding pattern of synchrony. In particular, this holds true for those patterns of synchrony that provide (the input networks of) the original network as a quotient of the fundamental network (compare to *Step* 4 of the step-by-step procedure to classify generic bifurcations in a semigroup network system in Section 6.1). We need the following technical result on subspaces of tensor product spaces.

Lemma 7.15. *Let $\mathbf{V}$ and $\mathbf{V}'$ be finite-dimensional real vector spaces and let $\mathbf{V}_1, \mathbf{V}_2 \subset \mathbf{V}$ and $\mathbf{V}'_1, \mathbf{V}'_2 \subset \mathbf{V}'$ be subspaces. Then $(\mathbf{V}_1 \otimes \mathbf{V}'_1) \cap (\mathbf{V}_2 \otimes \mathbf{V}'_2) = (\mathbf{V}_1 \cap \mathbf{V}_2) \otimes (\mathbf{V}'_1 \cap \mathbf{V}'_2)$ as a subspace of $\mathbf{V} \otimes \mathbf{V}'$.*

Proof. This result follows from the representation of a basis of the tensor product space in terms of bases of the components in Remark 7.1. Let $\{b_i\}_{i\in I}$ and $\{b'_j\}_{j\in J}$ be bases of $\mathbf{V}$ and $\mathbf{V}'$ respectively and let $I_1, I_2 \subset I$ and $J_1, J_2 \subset J$ be subsets such that

$$\mathbf{V}_1 = \langle b_i \mid i \in I_1 \rangle, \qquad \mathbf{V}_2 = \langle b_i \mid i \in I_2 \rangle$$
$$\mathbf{V}'_1 = \langle b'_j \mid j \in J_1 \rangle, \qquad \mathbf{V}'_2 = \langle b'_j \mid j \in J_2 \rangle.$$

Note that I_1 and I_2 can be constructed by completing a basis of $\mathbf{V}_1 \cap \mathbf{V}_2$ to bases of $\mathbf{V}_1$ and $\mathbf{V}_2$. Then the set of all basis elements of $\mathbf{V}_1$ and $\mathbf{V}_2$ is completed to a basis of $\mathbf{V}$. Accordingly we construct J, J_1, and J_2 for $\mathbf{V}'$ from a basis of $\mathbf{V}'_1 \cap \mathbf{V}'_2$. In particular, we obtain

$$\mathbf{V}_1 \cap \mathbf{V}_2 = \langle b_i \mid i \in I_1 \cap I_2 \rangle, \qquad \mathbf{V}'_1 \cap \mathbf{V}'_2 = \left\langle b'_j \mid j \in J_1 \cap J_2 \right\rangle.$$

Hence, using Remark 7.1 we see

$$(\mathbf{V}_1 \cap \mathbf{V}_2) \otimes (\mathbf{V}'_1 \cap \mathbf{V}'_2) = \left\langle b_i \otimes b'_j \mid i \in I_1 \cap I_2 \text{ and } j \in J_1 \cap J_2 \right\rangle.$$

On the other hand

$$\mathbf{V}_1 \otimes \mathbf{V}'_1 = \left\langle b_i \otimes b'_j \mid i \in I_1 \text{ and } j \in J_1 \right\rangle, \qquad (\mathbf{V}_2 \otimes \mathbf{V}'_2) = \left\langle b_i \otimes b'_j \mid i \in I_2 \text{ and } j \in J_2 \right\rangle.$$

Thus,

$$(\mathbf{V}_1 \otimes \mathbf{V}'_1) \cap (\mathbf{V}_2 \otimes \mathbf{V}'_2) = \left\langle b_i \otimes b'_j \mid i \in I_1 \cap I_2 \text{ and } j \in J_1 \cap J_2 \right\rangle$$

which completes the proof. □

As a corollary of Proposition 7.4 and Lemma 7.15 we obtain

Proposition 7.16. *Let $P = \{P_1, \dots, P_r\}$ be a balanced partition of the cells Σ and let Δ^1_P and Δ^D_P denote the corresponding robust synchrony subspaces in the case* **1D** *and* **DD** *respectively. Furthermore, let $Y \subset \mathbf{V}^1$ be an indecomposable component and $U \subset \mathbf{V}^\mathsf{D}$ such that $U \cong Y \otimes \langle w \rangle \cong Y$. Then*

$$\Delta^\mathsf{D}_P \cap U \cong (\Delta^1_P \otimes W) \cap (Y \otimes \langle w \rangle) = (\Delta^1_P \cap Y) \otimes \langle w \rangle.$$

7.2 Implications for bifurcations of steady states

In Theorem 7.12 we describe the relation between the algebraic structures of fundamental networks in the cases **1D** and **DD**. In particular, decomposing the regular representation into indecomposable subrepresentations in the case **1D** provides a decomposition in the case **DD** by choosing a basis for W. In this section we want to investigate how this allows us to reduce the investigation of bifurcations in fundamental networks with high-dimensional internal dynamics to that in fundamental networks with one-dimensional internal dynamics. In particular, we discuss how Theorem 7.12 can be used in *Step* 2 of the step-by-step procedure to classify generic bifurcations in a semigroup network system (see Section 6.1) to reduce the bifurcation problem in the case **DD** to one in the case **1D** (or some other lower dimension of the internal dynamics).

We begin by briefly recalling the setting of a bifurcation of steady states independent of the dimension of the internal phase space V. To that end, we assume that the fundamental network vector fields depend on real parameters $\lambda \in \mathbb{R}^l$ as in Section 3.2:

$$\Gamma_f(x, \lambda) = \begin{pmatrix} f(x_{\sigma_1\sigma_1}, \dots, x_{\sigma_n\sigma_1}, \lambda) \\ \vdots \\ f(x_{\sigma_1\sigma_n}, \dots, x_{\sigma_n\sigma_n}, \lambda) \end{pmatrix}. \tag{7.10}$$

Once again, without loss of generality we assume the bifurcation point to be the origin and the bifurcation to occur for $\lambda = 0$. That is, we assume

$$\Gamma_f(0, 0) = 0.$$

and $D_x\Gamma_f(0, 0)$ to have eigenvalues on the imaginary axis. We are interested in steady states and periodic solutions close to this bifurcation point for a generic smooth response function $f\colon \bigoplus_{\sigma\in\Sigma} V \times \mathbb{R}^l \to V$. Slightly more precisely a steady state bifurcation requires vanishing eigenvalues and a Hopf bifurcation requires purely imaginary eigenvalues of the linearization at the bifurcation point. In either case $D_x\Gamma_f(0,0)$ induces a decomposition of the right regular representation into subrepresentations given by its generalized kernel and reduced image or its center and hyperbolic subspaces respectively, i.e.

$$\bigoplus_{\sigma\in\Sigma} V = \ker_0(D_x\Gamma_f(0,0)) \oplus \operatorname{im}_0(D_x\Gamma_f(0,0)) \quad \text{or}$$
$$\bigoplus_{\sigma\in\Sigma} V = \mathcal{X}^c \oplus \mathcal{X}^h$$

in the notation of Section 3.2.

The bifurcation problem can be reduced to an equivalent one on the generalized kernel or the center subspace. In particular, branching steady state or periodic solutions of the original system lie on the center manifold over the corresponding subspace (see Section 3.5). Furthermore, they are fully characterized by symmetry of the subrepresentation. Hence, in order to classify all bifurcations of steady states that may occur in the fundamental network, one has to determine all possible subrepresentations of $\bigoplus_{\sigma\in\Sigma} V$ that can form center subspaces by decomposing the right regular representation space into indecomposable subrepresentations

$$\bigoplus_{\sigma\in\Sigma} V = W_1 \oplus \cdots \oplus W_m.$$

The generalized kernel or center subspace is isomorphic to the direct sum of a suitable collection of the components W_i. Theorem 7.12 shows that it is sufficient to determine this decomposition in the case **1D** – i.e. to decompose $\mathbf{V}^1$ – as each indecomposable component of $\mathbf{V}^{\mathsf{D}}$ is also an indecomposable component of $\mathbf{V}^1$. In particular, knowing the decomposition of $\mathbf{V}^1$ and if the dimension of the internal phase space W (in the case **DD**) is d, we immediately obtain the decomposition of $\mathbf{V}^{\mathsf{D}}$ in the form of d copies of the components of $\mathbf{V}^1$.

In order to properly describe generic generalized kernels and center subspaces we need the classification results in Theorem 5.11, Theorem 5.12, and Remark 5.13. We make the decomposition more precise by writing

$$\bigoplus_{\sigma\in\Sigma} V = V_1^{\mathrm{R}} \oplus \cdots \oplus V_{m_{\mathrm{R}}}^{\mathrm{R}} \oplus V_1^{\mathrm{C}} \oplus \cdots \oplus V_{m_{\mathrm{C}}}^{\mathrm{C}} \oplus V_1^{\mathrm{H}} \oplus \cdots \oplus V_{m_{\mathrm{H}}}^{\mathrm{H}},$$

where

$$V_i^{\mathrm{R}} \cong \left(\mathbf{X}_i^{\mathrm{R}}\right)^{s_i^{\mathrm{R}}}, \quad V_i^{\mathrm{C}} \cong \left(\mathbf{X}_i^{\mathrm{C}}\right)^{s_i^{\mathrm{C}}}, \quad V_i^{\mathrm{H}} \cong \left(\mathbf{X}_i^{\mathrm{H}}\right)^{s_i^{\mathrm{H}}}$$

are the isotypic components and $\mathbf{X}_i^{\mathrm{R}}, \mathbf{X}_i^{\mathrm{C}}$, and $\mathbf{X}_i^{\mathrm{H}}$ are indecomposable subrepresentations of real, complex, and quaternionic type respectively. Then we may determine, which configurations of indecomposable components are possible as generalized kernels or center subspaces for a generic l-parameter family of equivariant vector fields. In particular, let

$$U \cong \bigoplus_{i=1}^{m_{\mathrm{R}}} \left(\mathbf{X}_i^{\mathrm{R}}\right)^{\rho_i} \oplus \bigoplus_{i=1}^{m_{\mathrm{C}}} \left(\mathbf{X}_i^{\mathrm{C}}\right)^{\gamma_i} \oplus \bigoplus_{i=1}^{m_{\mathrm{H}}} \left(\mathbf{X}_i^{\mathrm{H}}\right)^{\iota_i}$$

with $0 \le \rho_i \le s_i^{\mathrm{R}}, 0 \le \gamma_i \le s_i^{\mathrm{C}}, 0 \le \iota_i \le s_i^{\mathrm{H}}$ for every i. Then U can only occur as a generalized kernel – i.e. $\ker_0(D_x\Gamma_f(0,0)) \cong U$ –, if

$$K_U = \sum_{i=1}^{m_{\mathrm{R}}} \rho_i + 2\cdot \sum_{i=1}^{m_{\mathrm{C}}} \gamma_i + 4\cdot \sum_{i=1}^{m_{\mathrm{H}}} \iota_i \le l, \tag{7.11}$$

and as a center subspace – i.e. $\mathcal{X}^c \cong U$ –, if

$$C_U = \sum_{i=1}^{m_{\mathrm{R}}} \lceil \rho_i/2 \rceil + \sum_{i=1}^{m_{\mathrm{C}}} \gamma_i + \sum_{i=1}^{m_{\mathrm{H}}} \iota_i \le l. \tag{7.12}$$

Here $\lceil a \rceil$ denotes the nearest larger or equal integer. The conditions (7.11) and (7.12) have simpler interpretations for the case of one-parameter bifurcations (see Remark 5.13).

In the upcoming subsections we investigate the interplay of Theorem 7.12 with the classification of generalized kernels and center subspaces in generic bifurcation problems. This allows to relate generic bifurcations of a fixed fundamental network with high-dimensional internal dynamics to those of the same network with one-dimensional internal dynamics. We begin with a discussion of the 1-parameter case, before before turning to general l-parameter families.

7.2.1 Generic 1-parameter steady state bifurcations

When focusing on steady state bifurcations, we want to characterize solutions to

$$\Gamma_f(x, \lambda) = 0$$

close to the bifurcation point $(x_0, \lambda_0) = (0, 0)$. As we mentioned before, Lyapunov-Schmidt reduction and center manifold reduction allow us to reduce the investigation of generic steady state bifurcations to a generic equivariant bifurcation problem on the generalized kernel $\ker_0(D_x\Gamma_f(0,0))$. From (7.11), we know that this center subspace generically is an absolutely indecomposable subrepresentation.

Let us turn to the case **DD**, i.e. to a bifurcation problem on $\mathbf{V}^{\mathrm{D}}$. Applying Theorem 7.12 (especially (7.9b)), we obtain that the generalized kernel in a given network in the case **DD** is isomorphic as a subrepresentation to one of the indecomposable subrepresentations one computes for $\mathbf{V}^1$ in the case **1D**. That is

$$\ker_0(D_\omega\Gamma_f(0,0)) \cong Y_i \subset \mathbf{V}^1$$

in the notation of Theorem 7.12. As the dynamics – in particular the generic steady state bifurcations – on these subrepresentations is entirely classified by symmetry, the reduced bifurcation problem in the case is equivalent to one on the subrepresentation Y_i. Hence, the generic steady state bifurcations in the case **DD** occur generically also in the case **1D**. On the other hand, as every component Y_i can occur as a generalized kernel $\ker_0(D_\omega\Gamma_f(0,0))$ in this way, any generic steady state bifurcation in the case **1D** occurs generically in the case **DD** as well. Summarizing, we have shown

Theorem 7.17. *The generic 1-parameter steady state bifurcations in a fundamental network with d-dimensional internal dynamics are qualitatively the same as those for the same network with 1-dimensional internal dynamics in the sense that the reduced bifurcation problems are equivalent in both cases.*

We cannot expect a more precise comparative result. The generalized kernels are the same in both cases so that the reduced bifurcation problems are the same. However, the full system has different dimensions and requires different coordinate systems. Therefore, the branching solutions for the full systems in general cannot be ‘equal’ in a stricter sense.

7.2.2 Generic 1-parameter Hopf bifurcations

Similar to Section 7.2.1, we investigate Hopf bifurcations in a generic 1-parameter family of fundamental network vector fields. In order to do so, we have to investigate possible center subspaces corresponding to non-vanishing purely imaginary eigenvalues. As in Remark 5.13, in combination with condition (7.12) only three cases can occur generically:

$$\mathcal{X}^c \cong \mathbf{X}_i^{\mathrm{C}}, \quad \mathcal{X}^c \cong \mathbf{X}_i^{\mathrm{H}}, \quad \mathcal{X}^c \cong \left(\mathbf{X}_i^{\mathrm{R}}\right)^2.$$

That is, either $\mathcal{X}^c$ is isomorphic to precisely one indecomposable component of complex or of quaternionic type or it is the direct sum of two isomorphic components of real type. In the case

DD Theorem 7.12 shows that the existence of two isomorphic components of real type in $\mathbf{V}^{\mathrm{D}}$ can occur in two different situations. Either $\mathbf{V}^1$ contains two isomorphic components of real type – i.e. $\mathcal{X}^c \cong Y_i^2 \subset \mathbf{V}^1$ in the notation of Theorem 7.12 – or $d \geq 2$ and we obtain two copies of the same component of $\mathbf{V}^1$ due to the high-dimensional internal dynamics – i.e. $\mathcal{X}^c \cong Y_i^2 \not\subset \mathbf{V}^1$. Moreover, the latter choice is the only center subspace for a generic 1-parameter Hopf bifurcation in the case **DD** that does not occur in the case **1D**. The other three cases are possible independent of d.

The solutions to the reduced bifurcation problem on $\mathcal{X}^c$ are entirely classified by symmetry. In particular, in the cases that are independent of d the reduced bifurcation problem is equivalent for each choice of d. The bifurcations are qualitatively the same as in the case **1D**. Furthermore, note that these cases describe all possible center subspaces in a generic 1-parameter Hopf bifurcation in the case **1D**. Hence, all generic 1-parameter Hopf bifurcations in the case **1D** can also be observed generically in the case **DD**. Conversely, the center subspace that is due to the high-dimensional internal dynamics – i.e. $\mathcal{X}^c \cong Y_i^2 \not\subset \mathbf{V}^1$ – can only occur as a generic center subspace for $d \geq 2$. Hence, in general there is no equivalent reduced bifurcation problem in the case **1D** and the corresponding Hopf bifurcations can only be observed in the case **DD**. The discussion of this subsection can be summarized as

Theorem 7.18. *(i) All center subspaces in generic 1-parameter Hopf bifurcations in the case* **1D** *are generic as center subspaces in a 1-parameter Hopf bifurcation in the case* **DD** *for any d. The branching periodic solutions corresponding to one center subspace are qualitatively the same for all values of d in the sense that the reduced bifurcation problems are equivalent.*

(ii) Let $\mathbf{V}^1 = Y_1 \oplus \cdots \oplus Y_s$ be a decomposition into indecomposable subrepresentations and assume Y_i to be of real type such that $Y_i \ncong Y_j$ for all $j \neq i$. If the internal dynamics is at least two-dimensional, i.e. $d \geq 2$, Y_i yields a center subspace $\mathcal{X}^c \cong Y_i^2$ of a generic 1-parameter Hopf bifurcation in the case **DD**. *The corresponding branching periodic solutions cannot be observed in the case* **1D**. *All remaining generic 1-parameter Hopf bifurcations in the case* **DD** *are as described in (i).*

Remark 7.19. Note that in both situations the indecomposable component of $\mathbf{V}^1$ can be high-dimensional due to symmetry. Hence, in general Theorem 7.18 does not describe a standard Hopf bifurcation with 2-dimensional center manifold. △

Corollary 7.20. *Assume that the specific network structure forces the fundamental network to decompose into only components of real type that are pairwise non-isomorphic. Then Hopf bifurcations in generic 1-parameter families are only possible in networks with internal dynamics of dimension greater or equal to 2.*

Remark 7.21. In particular both conditions of Corollary 7.20 hold true if the network structure forces the linear admissible maps to only have real eigenvalues. An example of this phenomenon is the class of feedforward networks that we investigate in Part III. △

7.2.3 Generic l-parameter bifurcations

The situation for l-parameter bifurcations is a lot more involved than in Sections 7.2.1 and 7.2.2. Similar precise statements relating the case **DD** to **1D** are not possible in full generality, as conditions (7.11) and (7.12) allow for greater flexibility in the composition of generalized kernel or the center subspace the larger the value of l is. Nevertheless, the underlying mechanism that made the two different characterizations in Theorem 7.18 possible, applies to l-parameter bifurcations (with $l > 1$) as well. In the case **DD** the representation space $\mathbf{V}^{\mathrm{D}}$ decomposes into the indecomposable subrepresentations of $\mathbf{V}^1$, each occurring d times. These subrepresentations can be components of generalized kernels or center subspaces. Hence, there are potentially numerous possibilities to find suitable combinations of components that satisfy (7.11) and (7.12). Nonetheless, any combination of subrepresentations that occurs in a generic l-parameter bifurcation in the case **1D** – i.e. one that does not make use of extra copies – also occurs as a generalized kernel or center subspace in a generic l-parameter bifurcation in the case **DD**. Once again, the reduced bifurcation problems are equivalent due to symmetry. They can be seen as the ones that are inherent to the network structure and independent of the internal dynamics. On the other hand, in general a generalized kernel or center subspace in a generic l-parameter bifurcation in the case **DD** with $d \geq 2$ contains multiple copies of the same indecomposable component of $\mathbf{V}^1$. Then there is no equivalent reduced bifurcation problem in the case **1D**. We summarize these results as

Theorem 7.22. *Generic l-parameter bifurcations in a fundamental network with 1-dimensional internal dynamics are also generic in the same network with d-dimensional internal dynamics.*

More generally, Theorem 7.22 follows almost directly from Corollary 7.14. The total phase space $\mathbf{V}^1$ is a subrepresentation of $\mathbf{V}^\mathrm{D}$. Hence, any combination of indecomposable components of $\mathbf{V}^1$ that make up a generalized kernel in a generic l-parameter bifurcation in the case **1D** also occur in a generic l-parameter bifurcation problem in the case **DD**. This yields the previous result. Even more so, it can be generalized to compare bifurcations in the same network with internal dynamics of arbitrary dimension. If $\dim W^1 \leq \dim W^2$ the total phase space $\bigoplus_{\sigma\in\Sigma} W^1$ is isomorphic to a subrepresentation of $\bigoplus_{\sigma\in\Sigma} W^2$. Hence, every generalized kernel or center subspace in a generic l-parameter bifurcation in $\bigoplus_{\sigma\in\Sigma} W^1$ also occurs as a generalized kernel or center subspace in a generic l-parameter bifurcation in $\bigoplus_{\sigma\in\Sigma} W^2$.

Theorem 7.23. *Generic l-parameter bifurcations in a fundamental network with d_1-dimensional internal dynamics are also generic in the same network with d_2-dimensional internal dynamics, whenever $d_1 \leq d_2$.*

On the other hand, for a fixed number of parameters l conditions (7.11) and (7.12) impose restrictions on the maximal number of indecomposable components of $\mathbf{V}^1$ that can occur as a generalized kernel or as a center subspace in a generic l-parameter bifurcation for any value of d. More precisely the generalized kernel can at most be composed of l components of real type, of $\lfloor l/2 \rfloor$ components of complex type, or of $\lfloor l/4 \rfloor$ components of quaternionic type. Here $\lfloor a \rfloor$ denotes the nearest smaller or equal integer. Likewise, the center subspace can at most be composed of $2l$ components of real type, of l components of complex type, or of l components of quaternionic type. In particular, the total number of indecomposable components is always less than or equal to l for generalized kernels and less than or equal to $2l$ for center subspaces. Recall that increasing the dimension of the internal dynamics d yields additional copies of the indecomposable components of $\mathbf{V}^1$ in the decomposition of $\mathbf{V}^\mathrm{D}$. In particular, we find all possible combinations of l or $2l$ indecomposable components in the case that $d = l$ or $d = 2l$ respectively. Increasing d beyond these critical values does not provide any further solutions to the combinatorial problems (7.11) and (7.12). As a result, all possible generalized kernels in a generic l-parameter bifurcation problem with internal dynamics of dimension $d' > l$ can also be observed in the case $d = l$. Likewise, all possible center subspaces in a generic l-parameter bifurcation problem with internal dynamics of dimension $d' > 2l$ can also be observed in the case $d = 2l$. Once again, the reduced bifurcation problems are therefore equivalent to those in the cases $d = l$ and $d = 2l$ respectively. In combination with Theorem 7.23 we obtain

Theorem 7.24. *(i) All generic l-parameter steady state bifurcations that can occur in a fundamental network can be investigated in the case of an internal phase space of dimension $d = l$.*

(ii) All generic l-parameter Hopf bifurcations that can occur in a fundamental network can be investigated in the case of an internal phase space of dimension $d = 2l$.

Remark 7.25. The minimal values of the dimension of the internal phase space d stated in Theorem 7.24 are optimal in the sense that there is a combination of indecomposable subrepresentations of $\mathbf{V}^1$ that can only occur in a generic l-parameter bifurcation problem if $d \geq l$ or $d \geq 2l$ respectively. To that end, let

$$\Delta_0 = \left\{ (v_\sigma)_{\sigma\in\Sigma} \in \mathbf{V}^1 \,\middle|\, v_\sigma = v_\tau, \quad \text{for all} \quad \sigma, \tau \in \Sigma \right\} \subset \mathbf{V}^1$$

be the fully synchronous subspace in the case **1D**. In particular, Δ_0 is a (not necessarily complementable) subrepresentation on which each representation map A^1_σ acts as the identity. Assume $Y \subset \mathbf{V}^1$ is a subrepresentation with $Y \cong \Delta_0$. Then all A^1_σ act as the identity on Y as well. That is, for all $y = (y_\sigma)_{\sigma\in\Sigma} \in Y$ we have $(A^1_\tau y)_\sigma = y_{\sigma\tau} = y_\sigma$ for all $\sigma, \tau \in \Sigma$ by definition of the right regular representation. In particular, for $\sigma = \mathrm{Id}$ we see

$$y_{\mathrm{Id}} = y_\sigma$$

for all $\sigma \in \Sigma$. As Δ_0 is one-dimensional, we obtain $y \in \Delta_0$ and therefore $Y = \Delta_0$. In particular, there is no subrepresentation in $\mathbf{V}^1$ that is isomorphic but not equal to Δ_0. This observation depends crucially on the fact that the internal dynamics are one-dimensional. In general, there is an indecomposable component of $\mathbf{V}^1$ that contains the fully synchronous subspace $Y \supset \Delta_0$. If a second subrepresentation $Y' \subset \mathbf{V}^1$ is isomorphic to Y the consideration above implies that Y and Y' contain Δ_0. Hence, $Y \cap Y' \supset \Delta_0 \neq \emptyset$.

As a result, any decomposition of $\mathbf{V}^1$ into indecomposable components contains precisely one component isomorphic to Y where $\Delta_0 \subset Y$. Consequently, in the case of d-dimensional internal dynamics there is a subrepresentation $U \subset \mathbf{V}^{\mathrm{D}}$ with $U \cong Y^l$ only if $d \geq l$ by Theorem 7.12. Likewise, a subrepresentation $U \subset \mathbf{V}^{\mathrm{D}}$ with $U \cong Y^{2l}$ exists only if $d \geq 2l$. △

Remark 7.26. In the case of 1-parameter bifurcations Theorem 7.24 shows that all generic steady state bifurcations can be observed in the network with 1-dimensional internal dynamics and all Hopf bifurcations can be observed in the case of 2-dimensional internal dynamics. This matches the results in Sections 7.2.1 and 7.2.2. △

7.2.4 Beyond qualitative statements using center manifold reduction

Sections 7.2.1 to 7.2.3 describe how to determine qualitative bifurcations in homogeneous coupled cell systems with (possibly) high-dimensional internal dynamics. In particular, how (parts of) the branching pattern in the case **DD** can be observed in the case **1D**. The reason why the restriction to qualitative statements needs to be made is the fact that the relation between the two cases is made in terms of the reduced bifurcation problems. The reduction methods require coordinate changes so that whatever information is stored in the precise choice of coordinates is lost when applying Theorems 7.17, 7.18 and 7.22. Most importantly, in the investigation of network dynamics this includes the possibility to distinguish individual cells from the coordinates of the total phase space variables. However, Theorem 7.12 allows for slightly more precise statements about the case **DD**, if more knowledge about bifurcations in the case **1D** is available. We can explicitly construct bifurcating branches in the case **DD** from those in the case **1D**, capturing the spirit of Theorem 7.22, without losing all information about each cell's behavior.

To that end, we make use of some of the technicalities of the center manifold reduction from Section 3.5 from where we also reuse some notation. Recall that we investigate the extended system

$$\dot{\underline{x}} = \begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} \Gamma_f(x, \lambda) \\ 0 \end{pmatrix} = \underline{\Gamma}_f(\underline{x}) \tag{7.13}$$

as in (3.20) in order to apply the center manifold reduction to bifurcation problems. Under the bifurcation assumption adapted to the extended system it admits a center manifold

$$M^c = \left\{ Q'(x^c, \lambda) + \underline{\psi}(Q'(x^c, \lambda)) \mid (x^c, \lambda) \in \mathcal{X}^c \times \Lambda \right\} \subset \bigoplus_{\sigma\in\Sigma} V \times \mathbb{R}^l \tag{7.14}$$

as in Corollary 3.40 which contains all branching solutions near the bifurcation point. In a slight abuse of notation we have denoted the subrepresentation that forms the corresponding generalized kernel or center subspace by $\mathcal{X}^c$ without distinction. In particular, the dynamics of a generic system of the form (7.13) restricted to the center manifold is bijectively conjugate to a generic system of the form

$$\begin{aligned} \dot{x}^c &= r(x^c, \lambda), \\ \dot{\lambda} &= 0 \end{aligned} \tag{7.15}$$

on $\mathcal{X}^c \times \mathbb{R}^l$, where

(i) $r(0,0) = 0$.

(ii) The center subspace of $D_{x^c} r(0,0)$ is the full space $\mathcal{X}^c$.

(iii) It is equivariant with respect to the monoid representation restricted to $\mathcal{X}^c$.

The conjugation is realized by the maps $P(x,\lambda) = (P^c(x), \lambda)$ and $\underline{\psi} \circ Q'$, where $P^c \colon \bigoplus_{\sigma \in \Sigma} V \to \mathcal{X}^c$ is an equivariant projection.

Assume that we know the branching behavior of each cell in a branch of a generic bifurcation of steady states **1D**. That is, we have a smooth curve of steady states or of initial conditions for periodic solutions $(v_\sigma(\lambda))_{\sigma \in \Sigma}$ for small absolute values of λ. We denote the subrepresentation that forms the corresponding generalized kernel or center subspace by $\mathcal{X}^c_1 \subset \mathbf{V}^1$. Due to the conjugation of the dynamics on the center manifold and the reduced system (7.15), we obtain that

$$P^c_1((v_\sigma(\lambda))_{\sigma \in \Sigma})$$

– where P^c_1 is the projection onto the generalized kernel or center subspace $\mathcal{X}^c_1$ in the case **1D** – is the branching solution of a generic reduced bifurcation problem corresponding to $(v_\sigma(\lambda))_{\sigma \in \Sigma}$. As the center subspace in general is a proper subspace, not every cell's coordinate entry $v_\sigma(\lambda)$ can be found in the projected solution branch. For example the projection might map coordinates to zero or sum multiple coordinate entries together. Nevertheless, as long as the projection P^c_1 is known, this method provides a method to represent the qualitative branching solutions of the reduced system while respecting the coordinates that are chosen according to the cells of the network. This projection operator, however, is often computed while classifying the generic bifurcations in the case **1D** as a byproduct.

Furthermore, Theorem 7.22 shows that any bifurcating branch in the case **1D** occurs generically in the same network in the case **DD** as well. Theorem 7.12 allows us to transform a generic **1D**-branch into a generic **DD**-branch. In particular, there is a generic l-parameter bifurcation problem in $\mathbf{V}^{\mathsf{D}}$ whose generalized kernel or center subspace is isomorphic to $\mathcal{X}^c_1$ in the case **1D**. Furthermore, from (7.9a) we see that this generalized kernel or center subspace $\mathcal{X}^c_{\mathsf{D}} \subset \mathbf{V}^{\mathsf{D}}$ is of the form

$$\mathcal{X}^c_{\mathsf{D}} \cong \mathcal{X}^c_1 \otimes \langle w \rangle, \tag{7.16}$$

where $w \in W \setminus \{0\}$. Denote the equivariant isomorphism in (7.16) by $\Psi \colon \mathcal{X}^c_1 \otimes \langle w \rangle \to \mathcal{X}^c_{\mathsf{D}}$. Consequently, the branch $(v_\sigma(\lambda))_{\sigma \in \Sigma}$ can be represented as a generic branch on the center subspace $\mathcal{X}^c_{\mathsf{D}}$ as

$$\Psi(P^c_1(v_\sigma(\lambda))_{\sigma \in \Sigma} \otimes w).$$

In more general terms, there exists a direction $w \in W \setminus \{0\}$ such that the generic bifurcation pattern – that is all generic bifurcating solutions – restricted to the center subspace in the 1-dimensional case is reflected in the d-dimensional case, where internal dynamics is restricted to this direction w. This interpretation, however, is only fully accurate in the case that Ψ is the identity – after identifying notations. In general it yields only qualitatively the same bifurcation diagram in the case

DD. Nevertheless, as the coordinates for the center subspace in the case **DD** reflect the cells of the network, we can read off cell-by-cell information from this representation. Finally, using the conjugacy between center manifold and reduced system once more, this time in the case **DD**, we find the representation of the branch $(v_\sigma(\lambda))_{\sigma\in\Sigma}$ in the center manifold of the **DD**-system as

$$\underline{\psi}^{\mathsf{D}}\left(Q'_{\mathsf{D}}\left(\Psi(P^c_1(v_\sigma(\lambda))_{\sigma\in\Sigma}\otimes w),\lambda\right)\right).$$

Due to Theorem 7.22 this branch occurs in a generic bifurcation problem with generalized kernel or center subspace isomorphic to $\mathcal{X}^c_1\otimes\langle w\rangle$.

In theory this procedure provides a mechanism to translate generic branching solutions in the case **1D** to generic solutions in the case **DD**. As long as information about the maps $P^c_1, Q'_{\mathsf{D}}, \underline{\psi}^{\mathsf{D}}$ and Ψ is available it also transforms information about the branching behavior of each individual cell. However, this latter part is not to be expected in general. The results in this chapter aim at simplifying the investigations of generic steady state bifurcations in the case **DD** by restricting to the case **1D**. In particular, we do not want to determine a generic center manifold in the high-dimensional case. As a result knowledge about these maps is not available in general. Nevertheless, the observations in this subsection provide a theoretical tool to relate generic branching solutions in the case **DD** to those of the case **1D**. In Chapter 11 we investigate how additional structure in the network – in particular feedforward structure – provides information about the maps $P^c_1, Q'_{\mathsf{D}}, \underline{\psi}^{\mathsf{D}}$ and Ψ without explicitly computing them. This suffices to exploit this mechanism to characterize generic branching solutions in the case **DD** from those in the case **1D** including the behavior of each cell without computing the center manifolds.

Remark 7.27. The method presented in this section can also be used in the spirit of Theorem 7.24 to translate a bifurcating branch of steady state or periodic solutions in a fundamental network with d_1-dimensional internal dynamics into one for the same network with d_2-dimensional internal dynamics whenever the corresponding generalized kernel or center subspace occurs in both cases generically. In particular, when a branch exhibits a robust pattern of synchrony – i.e. coordinate entries corresponding to a balanced partition of the cells coincide – then the representation of that branch exhibits the same synchrony for internal dynamics of any dimensions for which it exists generically. This follows from Proposition 7.16. △

Part III

Feedforward networks

Contents

Chapter 8

Four equivalent definitions

In Part III of this thesis we apply the methods introduced and derived in Part II to the class of *feedforward networks*. This structural property of a network is characterized by the absence of any feedback indicating some sort of ordering or regularity. This has multiple simplifying effects. As a result feedforward networks are often found in the sciences and in particular in modeling thereof. For example in artificial neural networks data is fed into a group nodes of the networks. After processing, they pass it on to some other group which, in turn, process it and feed the information to other cells. In doing so, a cell never receives inputs twice. Thus, the information gets processed in 'forward' direction only until it reaches some final cells which generate the output of the neural network. Furthermore, feedforward networks have also been studied in network dynamics where they exhibit interesting phenomena such as amplification effects in bifurcations. Usually these investigations are restricted to feedforward *chains* or *layered* feedforward networks. Here we consider a more general framework. It includes feedforward chains and layered feedforward networks as an example. To that end, we define feedforward structure in a very general formulation in this chapter. We explore the implications of this definition on semigroup networks by relating it to algebraic properties of the corresponding monoid of input maps as well as to order theoretic properties of the network cells. We then use these connections to provide multiple equivalent definitions of feedforward networks. In Chapter 9 we characterize the algebraic properties of the fundamental network of a feedforward network as well as the hidden symmetries it induces. In particular, we prove that the regular representation of the monoid can be decomposed into indecomposable subrepresentations by means of an algorithm. We also exploit the different definitions to thoroughly investigate generic 1-parameter steady state bifurcations in Chapters 10 and 11 (for one-dimensional and high-dimensional internal dynamics). We show that in both cases these networks exhibit an amplification effect similar – but more complex – to that observed before. In particular, branching solutions can be determined by an inductive procedure that essentially only depends on the network structure. Interestingly, different regions of the space of Taylor coefficients of the response function may exhibit different patterns of branching solutions. Finally, we illustrate the results in an example in Chapter 12. For background information and further motivation see the introduction Chapter 1.

8.1 Loop-free networks

We begin with the first – highly general – definition of feedforward structure and some algebraic observations. We remain in the setting of semigroup networks as introduced in Chapter 3. This chapter is essentially Section 2 of our preprint [84].

Definition 8.1. *A homogeneous coupled cell network is called a* feedforward network *if it has no directed loops involving 2 or more nodes (consisting of arrows not necessarily of the same type). Put differently, a network is a feedforward network if the only directed loops are self-loops.*

Remark 8.2. The name *feedforward network* is natural in the sense that all the feedback a cell can receive is via a self-arrow. This definition can naturally be extended to networks outside of the semigroup network framework. △

Remark 8.3. Note that the property of being a feedforward network does not change if the set of input maps is completed to a monoid. In particular, this property can be checked by looking only at the generators of Σ. For, if there is a directed loop of 2 or more nodes consisting of generator-arrows, then this loop still exists if we add concatenations of these arrows to the network.

Conversely, suppose we have a directed loop of any arrows. Then by expressing these arrows as concatenations of generators, we get a directed loop consisting of only generators (or possibly multiple directed loops, one of which connects all the nodes involved). In particular, adding all concatenations of arrow-types to a network (i.e. 'completing the monoid') does not change whether or not the network is a feedforward network. △

In the case that the input maps form a monoid, we relate the structural property defined above to an algebraic property of the corresponding monoid of input maps of the network.

Definition 8.4 (p. 7 in [105])**.** *A monoid Θ is called* L-trivial *if the sets $\Theta\theta = \{\kappa\theta \mid \kappa \in \Theta\}$ are different for all $\theta \in \Theta$. In other words, if for $\theta, \theta' \in \Theta$ we have that $\Theta\theta = \Theta\theta'$ implies $\theta = \theta'$.*

A useful lemma in determining if a monoid is L-trivial is the following.

Lemma 8.5. *Let Θ be a (not necessarily L-trivial) monoid. For $\theta, \theta' \in \Theta$ we have $\Theta\theta \subset \Theta\theta'$ if and only if there exists an element $\kappa \in \Theta$ such that $\theta = \kappa\theta'$. Hence, we have $\Theta\theta = \Theta\theta'$ if and only if there exist $\kappa, \iota \in \Theta$ such that $\theta = \kappa\theta'$ and $\theta' = \iota\theta$.*

Proof. If $\Theta\theta \subset \Theta\theta'$ holds then we have $\theta = \mathrm{Id}\,\theta \in \Theta\theta \subset \Theta\theta'$. Hence we see that $\theta = \kappa\theta'$ for some $\kappa \in \Theta$. Conversely, from $\theta = \kappa\theta'$ we get $\Theta\theta = \Theta\kappa\theta' \subset \Theta\theta'$, where we have used that $\Theta\kappa \subset \Theta$. This proves the first part of the lemma. The second follows immediately from this equivalence. □

The following lemmas show that the monoid of input maps of a feedforward network is L-trivial. For the sake of convenience, and motivated by Remark 8.3 above, we assume from now on that the input maps of any network considered form a monoid. Then the condition on the input structure of a feedforward network reads $\sigma(p) = q$ and $\tau(q) = p$ together imply $p = q$ in terms of input maps. Strictly speaking, this only excludes loops involving exactly 2 nodes. However, by concatenation of arrows – or equivalently of input maps – we see that a loop consisting of arbitrary many cells induces loops of any two cells involved. Hence, it is excluded by this definition as well.

Lemma 8.6. *A (semigroup) network $\mathcal{N} = (C, \Sigma)$ is a feedforward network if and only if for all nodes $p, q \in C$ the sets $\Sigma(p) = \{\sigma(p) \mid \sigma \in \Sigma\}$ are different, i.e. we have $\Sigma(p) = \Sigma(q)$ implies $p = q$.*

Proof. Let p and q be two different nodes such that $\Sigma(p) = \Sigma(q)$. Then $p = \mathrm{Id}(p) \in \Sigma(p) = \Sigma(q)$. Hence, there exists an element $\sigma \in \Sigma$ such that $p = \sigma(q)$. Analogously, we find an element $\tau \in \Sigma$ such that $\tau(p) = q$ and we conclude that there is a directed loop in the network connecting the two nodes p and q. This proves that the network is not a feedforward network whenever $\Sigma(p) = \Sigma(q)$ for some $p \neq q$.

Conversely, suppose the network is not a feedforward network, so that there is a directed loop connecting 2 or more nodes. By picking two different nodes in the loop and by concatenating arrows if necessary, we see that there exist nodes $p, q \in C$, $p \neq q$ and elements $\sigma, \tau \in \Sigma$ such that $p = \sigma(q)$ and $q = \tau(p)$. It follows that $\Sigma(p) = \Sigma(\sigma(q)) = \Sigma\sigma(q) \subset \Sigma(q)$, and likewise $\Sigma(q) \subset \Sigma(p)$. Hence, we have $\Sigma(p) = \Sigma(q)$ for some nodes $p \neq q$. This proves the lemma. □

As the nodes in the fundamental network are given by the monoid Σ (with the action of the input maps given by left-multiplication), we immediately find:

Corollary 8.7. *A fundamental network is a feedforward network if and only if the underlying monoid is L-trivial.*

Remark 8.8. Note that L-triviality can easily be checked from the multiplication table of a monoid Θ; the inequality $\Theta\theta \neq \Theta\theta'$ for $\theta \neq \theta'$ simply implies different columns in the multiplication table have a different set of monoid elements appearing. Likewise, for a general network the inequality $\Sigma(p) \neq \Sigma(q)$ for different nodes $p \neq q$ simply means the set of nodes that p depends on is different from those that q depends on (this includes the self-loop given by $\mathrm{Id} \in \Sigma$). △

Next, we investigate how feedforward structure behaves under constructing the fundamental network.

Lemma 8.9. *A network is a feedforward network if and only if its fundamental network is a feedforward network.*

Proof. Suppose the fundamental network is not a feedforward network, so that we have $\Sigma\sigma = \Sigma\tau$ for some different $\sigma, \tau \in \Sigma$. As we have $\sigma \neq \tau$, there exists at least one node p such that $\sigma(p) \neq \tau(p)$. We therefore find $\Sigma(\sigma(p)) = \Sigma\sigma(p) = \Sigma\tau(p) = \Sigma(\tau(p))$ where $\sigma(p)$ and $\tau(p)$ are two different nodes. This proves that the original network is not a feedforward network.

Conversely, suppose the original network is not a feedforward network, and therefore has a directed loop consisting of multiple nodes. It follows that in particular, we have two different nodes p and q and monoid elements $\sigma, \tau \in \Sigma$ such that $\sigma(p) = q$ and $\tau(q) = p$. As Σ has only finitely many elements, we see that for some positive integers s and t we have $(\tau\sigma)^s = (\tau\sigma)^{s+t}$. We will focus our attention on the two monoid elements $\kappa = (\tau\sigma)^s$ and $\iota = \sigma(\tau\sigma)^s$. First of all, it clearly holds that

$$\iota = \sigma\kappa,$$

so that $\Sigma\iota \subset \Sigma\kappa$ due to Lemma 8.5. Moreover, we find

$$[(\tau\sigma)^{t-1}\tau]\iota = (\tau\sigma)^{t-1}\tau\sigma(\tau\sigma)^s = (\tau\sigma)^{t+s} = (\tau\sigma)^s = \kappa$$

showing that $\Sigma\kappa \subset \Sigma\iota$. We conclude that $\Sigma\kappa = \Sigma\iota$. However, by construction we have $\kappa(p) = (\tau\sigma)^s(p) = p$, whereas we also find $\iota(p) = \sigma\kappa(p) = \sigma(p) = q$. As $p \neq q$ by assumption, we see that κ and ι are two different elements in the monoid. This shows that the fundamental network is not a feedforward network, proving the lemma. □

Remark 8.10. Note that in Lemma 8.9 we do not assume the original network to be a quotient of the fundamental network. In other words, we do not require the existence of a node in the network that 'feels' all the other nodes. △

Corollary 8.7 and Lemma 8.9 can be summarized as

Corollary 8.11. *A network is a feedforward network if and only if the underlying monoid of input maps Σ is L-trivial.*

Lastly, we show that feedforward structure behaves well under taking quotients.

Lemma 8.12. *The quotient of a feedforward network by a balanced partition is again a feedforward network.*

Proof. As is often the case with feedforward networks, it is easier to prove the contrapositive: if the quotient network is not a feedforward network, then neither is the original network. Let us write $[p]$ for the equivalence class of a node p in the original network under the relevant balanced partition. The assumption that the quotient network is not a feedforward network means that there exist nodes p and q such that $[p] \neq [q]$ and monoid elements $\sigma, \tau \in \Sigma$ such that $\sigma[p] = [q]$ and $\tau[q] = [p]$. As in the proof of Lemma 8.9, we may define $\kappa = (\tau\sigma)^s \in \Sigma$ and $\iota = \sigma(\tau\sigma)^s \in \Sigma$ for some large enough s and it follows that $\Sigma\kappa = \Sigma\iota$. Moreover, as κ sends the equivalence class $[p]$ to itself whereas ι sends $[p]$ to $[q] \neq [p]$, we have that $\kappa \neq \iota$. We conclude that Σ is not L-trivial. By Lemma 8.9 we see that the original network is not a feedforward network either. This proves the lemma. □

Now that we have established a 'visual' interpretation of the algebraic notion of L-triviality, we can start exploiting the technical consequences of this definition. We choose a labeling of the nodes $C = \{p_1, \dots p_N\}$ such that it holds that

$$\#\Sigma(p_1) \geq \#\Sigma(p_2) \geq \dots \geq \#\Sigma(p_N). \tag{8.1}$$

Recall that we denote the algebra of linear maps on the internal phase space of the coupled cell system by $\mathfrak{gl}(V)$.

Lemma 8.13. *Choosing the ordering of cells according to* (8.1) *for a feedforward network yields that any linear admissible map can be represented by an upper triangular matrix with entries in* $\mathfrak{gl}(V)$. *In particular, if* $V = \mathbb{R}$ *we identify* $\mathfrak{gl}(V) \cong \mathbb{R}$ *to see that the linear admissible maps can be represented by real upper triangular matrices.*

Proof. We claim that p_i can only be an element of $\Sigma(p_j)$ if $i \geq j$. Indeed, if we have $p_i \in \Sigma(p_j)$ then $\Sigma(p_i) \subset \Sigma\Sigma(p_j) = \Sigma(p_j)$. In particular then, we have $\#\Sigma(p_i) \leq \#\Sigma(p_j)$. If it also holds that $i < j$ then $\#\Sigma(p_i) \geq \#\Sigma(p_j)$, and so $\#\Sigma(p_i) = \#\Sigma(p_j)$. From $\Sigma(p_i) \subset \Sigma(p_j)$ it now follows that $\Sigma(p_i) = \Sigma(p_j)$, directly contradicting that the network is a feedforward network (note that $i < j$ implies in particular that $p_i \neq p_j$).

Now the set $\Sigma(p)$ consists of exactly those nodes that (the variable of) p depends on. We see that p_2 does not depend on p_1, p_3 does not depend on p_1 and p_2 etc. This shows that all linear admissible maps are indeed in upper triangular form with respect to this ordering. □

Remark 8.14. In Remark 8.20 we use a partial order on the nodes inherited from the network structure which induces upper triangularity of linear admissible maps as well. △

Lemma 8.13 has immediate consequences for generic 1-parameter Hopf bifurcations in feedforward networks. These can only occur, when the internal dynamics is at least 2-dimensional (compare to Section 7.2).

Theorem 8.15. *In a feedforward network a* 1*-parameter Hopf bifurcation can only occur if the internal dynamics is at least* 2*-dimensional.*

Proof. In order for a Hopf bifurcation to occur, a 1-parameter family of linear admissible maps has to have a pair of complex conjugate eigenvalues at the bifurcation point (see Section 3.2). In the case $V = \mathbb{R}$ all linear admissible maps are real upper triangular matrices. Their eigenvalues are the diagonal elements which are real. Hence, a Hopf bifurcation is not possible. On the other hand, when the internal dynamics is in $V \cong \mathbb{R}^d$ with $d \geq 2$, linear admissible maps are upper triangular with entries in $\mathfrak{gl}(V)$. Thus the eigenvalues of a linear admissible map are the union of the eigenvalues of all diagonal elements which are arbitrary elements in $\mathfrak{gl}(V)$. In particular, there are possible diagonal elements with complex eigenvalues so that Hopf bifurcation is generically possible. □

In the case $V = \mathbb{R}$ the linear admissible maps are real upper triangular matrices. These in turn have their eigenvalues on the diagonal. We investigate further algebraic and spectral properties in Chapter 9. First, we introduce the following definition that turns out to be useful for determining diagonal entries of linear admissible maps.

Definition 8.16. *Given a homogeneous coupled cell network (not necessarily of feedforward type), we define an equivalence relation* $\multimap$ *on the nodes as follows:* $p \multimap q$ *if*

$$\{\sigma \in \Sigma \mid \sigma(p) = p\} = \{\sigma \in \Sigma \mid \sigma(q) = q\}. \tag{8.2}$$

We denote the sets by $\mathcal{L}_p = \{\sigma \in \Sigma \mid \sigma(p) = p\}$. *If* $p \multimap q$ *then we say that* p *and* q *have the same* loop-type. *In a network, two nodes have the same loop-type if and only if they have the same self-loops (of the same colors given by* $\mathcal{L}_p$*).*

To define loop-types on an arbitrary network we do not require that the set of input maps Σ is a monoid. However, passing to the smallest monoid containing Σ might make the loop-type relation finer. Nevertheless, for feedforward networks this relation remains the same, as the next lemma shows.

Lemma 8.17. *In a feedforward network with monoid Σ generated by the elements $\sigma_1, \ldots, \sigma_m$, we have $p \multimap q$ if and only if*

$$\{i \in \{1, \ldots, m\} \mid \sigma_i(p) = p\} = \{i \in \{1, \ldots, m\} \mid \sigma_i(q) = q\}. \tag{8.3}$$

Proof. If p and q have the same loop-type, then they are fixed by the same elements of Σ. In particular then, they are fixed by the same generators σ_i. This proves that equation (8.3) holds.

Conversely, assume (8.3) holds true. Suppose we have a general element $\tau \in \Sigma$ such that $\tau(p) = p$. If τ equals the identity then we also have $\tau(q) = q$. Therefore, we may assume that τ equals a product of generators $\sigma_{i_1} \cdots \sigma_{i_k}$ for some $k \geq 1$. Defining $r = \sigma_{i_k}(p)$, we see that $\sigma_{i_1} \cdots \sigma_{i_{k-1}}(r) = \sigma_{i_1} \cdots \sigma_{i_k}(p) = p$. As we also have $r = \sigma_{i_k}(p)$, it follows from the feedforward structure of the network that $r = p$. We therefore find $p = \sigma_{i_k}(p)$ as well as $\sigma_{i_1} \cdots \sigma_{i_{k-1}}(p) = p$. Continuing this way, we find $\sigma_{i_1}(p) = \ldots = \sigma_{i_k}(p) = p$. By the assumption that (8.3) holds, it follows that $\sigma_{i_1}(q) = \ldots = \sigma_{i_k}(q) = q$, and so $\tau(q) = \sigma_{i_1} \cdots \sigma_{i_k}(q) = q$. Repeating this argument with the roles of p and q reversed, we find that indeed $p \multimap q$. This proves the lemma. □

Remark 8.18. The loop-type of a node p can easily be read off from the admissible vector fields of a network; it is given by those entries of the response function through which the variable x_p depends on x_p. △

8.2 A partial order in the network

Next, we take an order-theoretic perspective on networks. We introduce a preorder on the set of cells induced by the network structure and relate it to the structural features discussed in Section 8.1. Consider the preorder

$$p \trianglelefteq q \quad \Longleftrightarrow \quad \text{there is an arrow from } q \text{ to } p.$$

Using the input maps this means there exists an input map sending cell p to cell q:

$$p \trianglelefteq q \quad \Longleftrightarrow \quad \text{there exists } \sigma \in \Sigma \text{ with } \sigma(p) = q.$$

As every cell is coupled to itself via the internal dynamics the preorder is obviously reflexive. On the other hand if there is an arrow from cell q to cell p and an arrow from cell r to cell q the concatenated arrow goes from r to p. Concatenation of arrows is precisely the operation performed when the set of input maps is completed to a monoid. From now on, we assume this to be done which makes the preorder transitive as well. However, in the definition we do not exclude the possibility of an arrow from q to p and one from p to q for two different cells p and q. In that case we have $p \trianglelefteq q$ and $q \trianglelefteq p$ even though $p \neq q$. Thus, in general the preorder is no partial order. We encode the situation that $p \trianglelefteq q$ and $q \trianglelefteq p$ using $p \vartriangle q$. Furthermore, $p \trianglelefteq q$ but $p \neq q$ is denoted by $p \triangleleft q$.

Proposition 8.19. *The preorder $\trianglelefteq$ is a partial order if and only if the network is a feedforward network.*

Proof. Assume the network is not a feedforward network. Then there is a loop $\{p_1, \ldots, p_k\}$ with $k \geq 2$ in the network, meaning there are input maps $\sigma_1, \ldots, \sigma_k$ such that $\sigma_1(p_1) = p_k$ and $\sigma_i(p_i) = p_{i-1}$ for $2 \leq i \leq k$. For $\tau = \sigma_2 \cdots \sigma_k$ we have $\tau(p_k) = p_1$ which implies $p_k \trianglelefteq p_1$. On the other hand, we also have $p_1 \trianglelefteq p_k$ from $\sigma_1(p_1) = p_k$. As $p_1 \neq p_k$, this shows the preorder is no partial order.

Now assume that the preorder is not a partial order. Then there are cells $p \neq q$ with $p \trianglelefteq q$ and $q \trianglelefteq p$. This implies the existence of input maps σ and τ such that $\sigma(p) = q$ and $\tau(q) = p$. This is the same as an arrow from q to p and one from p to q making these cells a loop of size 2. Thus, the network is not a feedforward network. □

Remark 8.20. Note that we can use the partial order $\trianglelefteq$ to label the cells in such a way, that the admissible linear maps are upper triangular – compare to Lemma 8.13. We choose a labeling of the nodes $\{p_1, \ldots, p_N\}$ such that whenever $p_i \trianglelefteq p_j$ it also holds that $i \leq j$ – this is not unique as some elements may not be related by the partial order. By definition for an input $\sigma(p)$ of cell p it holds that $\sigma(p) \trianglerighteq p$ and therefore its index is greater than or equal to that of p. This shows that the linear admissible maps are upper triangular.

Conversely, given a labeling $\{p_1, \ldots, p_N\}$ such that all linear admissible maps are upper triangular. Suppose $p_i \trianglelefteq p_j$ and $p_j \trianglelefteq p_i$. Then there exist $\sigma, \tau \in \Sigma$ such that $\sigma(p_i) = p_j$ and $\tau(p_j) = p_i$. These maps describe the input structure that is reflected by the linear admissible maps. Hence, there exists such a map with non-zero values in the (i, j)-th as well as in the (j, i)-th entry. Due to the upper triangular structure, this can only be true for $i = j$ and therefore $p_i = p_j$. Hence $\trianglelefteq$ is a partial order. Summarizing, the upper triangular structure of linear admissible maps is equivalent to $\trianglelefteq$ being a partial order on C. We could have used this as a definition for feedforward networks as well. △

Note that, since the network contains only finitely many cells, there are well-defined maximal elements with respect to the partial order $\trianglelefteq$. By definition, these are cells that do not receive any inputs from other cells. This characterization has some immediate useful consequences that we collect here.

Lemma 8.21. *A cell $p \in C$ is maximal if and only if it is fixed by the entire monoid, i.e. $\sigma(p) = p$ for all $\sigma \in \Sigma$. In particular, a maximal cell receives all its inputs from itself.*

Proof. This follows almost directly from the definition of the partial order $\trianglelefteq$. We prove the statement by contraposition. Assume, $\sigma(p) \neq p$ for some $\sigma \in \Sigma$. Then there is an arrow from $\sigma(p)$ to p. This yields $\sigma(p) \triangleright p$ so that p is not maximal. On the other hand, if $q \triangleright p$ then there is an arrow from q to p and equivalently an input map $\sigma \in \Sigma$ with $\sigma(p) = q$. As $q \triangleright p$ implies $q \neq p$, this shows that p is not fixed by the entire monoid. □

Corollary 8.22. *From Lemma 8.21 we obtain $\mathcal{L}_p = \Sigma$ for all maximal $p \in C$ (compare to Definition 8.16). In particular, for p maximal, $p \multimapboth q$, if and only if q is maximal.*

The following three corollaries characterize the relation of the notions of paths (Definition 3.2) and blocks (Definition 3.34) to that of the partial order $\trianglelefteq$.

Corollary 8.23. *Stating the definition of a maximal element in terms of arrows in the network immediately proves that every cell p is either maximal itself or there is a path from a maximal cell $\overline{p}$ to p. Furthermore, it is obvious that $q \triangleright p$ implies that q is on some path from a maximal cell to p, i.e. there is a maximal cell $\overline{p}$ such that $q \in \omega \in \Omega_{\overline{p},p}$ (the set of paths from $\overline{p}$ to p as in Definition 3.2).*

Corollary 8.24. *There is an element $\kappa \in \Sigma$ mapping every cell $p \in C$ to a maximal one simultaneously, i.e. $\kappa(p)$ is maximal for all $p \in C$.*

Proof. Let $p_1 \in C$ be any cell. Due to Corollary 8.23 there is a maximal cell $\overline{p}_1$ such that there exists a path from $\overline{p}_1$ to p. In particular, there are input maps $\sigma_1, \ldots, \sigma_k$ such that $\sigma_k(\sigma_{k-1}(\ldots(\sigma_1(p_1)\ldots))) = \overline{p}_1$. Hence, setting $\kappa_1 = \sigma_k \cdots \sigma_1$, we have $\kappa_1(p_1) = \overline{p}_1$. If p_1 is maximal, we may choose $\kappa_1 = \mathrm{Id}$. Then choose $p_2 \in C \setminus \{p_1\}$ and construct κ_2 accordingly as the map such that $\kappa_2(\kappa_1(p_2)) = \overline{p}_2$ is maximal. As $\overline{p}_1$ is maximal, we have $\sigma(\overline{p}_1) = \overline{p}_1$ for all $\sigma \in \Sigma$. In particular, $\kappa_2(\overline{p}_1) = \kappa_2(\kappa_1(p_1)) = \overline{p}_1$. Thus, we see that $\kappa_2 \circ \kappa_1$ maps $p_1 \mapsto \overline{p}_1$ and $p_2 \mapsto \overline{p}_2$

which are both maximal. Continuing recursively, we construct a map κ_i for each $p_i \in C$ so that $(\kappa_i \circ \cdots \circ \kappa_1)(p_j) = \overline{p}_j$ is maximal for all $j \le i$. As there are only finitely many cells, setting

$$\kappa = \kappa_N \circ \cdots \circ \kappa_1$$

completes the proof. □

Corollary 8.25. *In the context of feedforward networks, a block $B \subset C$ is defined by the property that $p \in B$ implies $q \in B$ for all $q \rhd p$. This, in turn, yields that every block contains at least one maximal cell.*

In the remainder of this section, we investigate some properties of the preorder $\unlhd$ on the fundamental network. We do not assume it to be a partial order. Note that the preorder on the fundamental network boils down to a preorder on the monoid Σ, as the input maps Σ act on the cells Σ via multiplication from the left. That is,

$$\sigma \unlhd \tau \qquad \Longleftrightarrow \qquad \text{there exists } \sigma' \in \Sigma \text{ with } \sigma'\sigma = \tau.$$

This can be defined for any monoid and does not require L-triviality. As a network is feedforward if and only if its fundamental network is (see Lemma 8.9), we obtain

Corollary 8.26. *The preorder $\unlhd$ on the original network is a partial order if and only if the corresponding preorder on the fundamental network is a partial order.*

To understand the structure of a given monoid Θ, one often investigates the so called *Green's relations* of which we only introduce one at this point. That is, two elements $\theta, \theta' \in \Theta$ are L-related, denoted by $\theta L \theta'$, if and only if they generate the same left ideals in the monoid: $\Theta\theta = \Theta\theta'$. Recall from Definition 8.4 that the monoid is L-trivial if and only if this relation is equal to the identity relation. The L-relation can also be defined using the so called *Green's-L-preorder* $\le_L$ where

$$\theta \le_L \theta' \qquad \Longleftrightarrow \qquad \Theta\theta \subset \Theta\theta'.$$

Then

$$\theta L \theta' \qquad \Longleftrightarrow \qquad \theta \le_L \theta' \text{ and } \theta' \le_L \theta$$

We can easily see that Green's-L-preorder is the order dual to the preorder $\unlhd$. Note a similar result in Remark 2.7 in Berg et al. [18].

Lemma 8.27. *Green's-L-preorder is order dual to the preorder $\unlhd$.*

Proof. For two elements $\theta, \theta' \in \Theta$ assume $\theta \le_L \theta'$, i.e. $\Theta\theta \subset \Theta\theta'$. According to Lemma 8.5 this is equivalent to the existence of an element $\kappa \in \Theta$ such that $\kappa\theta' = \theta$. This, in turn, is equivalent to $\theta' \unlhd \theta$. □

Remark 8.28. It follows immediately from the definition of $\le_L$ that the monoid is L-trivial if and only if $\le_L$ is a partial order. Therefore, we could have used this formalism to prove the equivalence of the definitions of feedforward networks and that of the partial order $\unlhd$ for fundamental networks. △

We provide further results concerning least and greatest elements of a finite monoid Θ with respect to $\unlhd$ and bear in mind that these can directly be applied to fundamental networks. Therein, we call an element $\overline{\theta} \in \Theta$ a greatest element if $\overline{\theta} \unrhd \theta$ for all $\theta \in \Theta$ and $\overline{\theta} \unlhd \theta$ implies $\theta = \overline{\theta}$. Similarly, it is a least element if $\overline{\theta} \unlhd \theta$ for all $\theta \in \Theta$ and $\theta \unlhd \overline{\theta}$ implies $\theta = \overline{\theta}$. When $\unlhd$ is a partial order the second condition in both cases is redundant and we obtain the usual definition of greatest and least elements in a partially ordered set.

Lemma 8.29. *An element $z \in \Theta$ is a greatest element with respect to $\trianglelefteq$ if and only if it is a (two-sided) zero element in the monoid.*

Proof. Suppose $z \in \Theta$ is a greatest element with respect to $\trianglelefteq$ which means $z \trianglerighteq \theta$ for all θ and $z \trianglelefteq \theta$ implies $\theta = z$. Using the definition of the preorder, the second condition implies $\theta z = z$ for all $\theta \in \Theta$, as $\theta z \trianglerighteq z$ by definition. Therefore, z is a right zero element. Using the first condition we obtain for every $\theta \in \Theta$ there exists $\theta' \in \Theta$ such that $\theta'\theta = z$. In particular, for arbitrary $\theta \in \Theta$ there is $\theta' \in \Theta$ such that $\theta'(z\theta) = z$ and since z is a right zero element $z = \theta'(z\theta) = (\theta' z)\theta = z\theta$. Thus, z is also a left zero element.

On the other hand, let $z \in \Sigma$ be a zero element. We immediately obtain $z\theta = z$ for all $\theta \in \Theta$ and therefore $z \trianglerighteq \theta$ for all $\theta \in \Theta$. Next, assume there is an element $\theta \in \Theta$ such that $z \trianglelefteq \theta$. Then there exists $\theta' \in \Theta$ such that $\theta' z = \theta$. As z is a zero element this implies $z = \theta$. Therefore z is a greatest element. □

Remark 8.30. As a monoid can have at most one zero element, we get that greatest elements with respect to $\trianglelefteq$ in the fundamental network are unique, even if $\trianglelefteq$ is not a partial order. △

The following lemma serves as a reminder that left- and right-inverses are equal – and therefore two-sided inverses – in a finite monoid.

Lemma 8.31. *If the element θ of the finite monoid Θ has a left-(right-)inverse θ', then θ' is the unique two-sided-inverse of θ.*

Proof. Assume $\theta'\theta = \mathrm{Id}$. We deduce for $\kappa, \iota \in \Theta$ such that $\theta\kappa = \theta\iota$

$$\theta\kappa = \theta\iota \quad \Longrightarrow \quad \theta'(\theta\kappa) = \theta'(\theta\iota) \quad \Longrightarrow \quad \kappa = \iota.$$

Hence, the map $\kappa \mapsto \theta\kappa$ is injective and by finiteness of Θ also bijective. Therefore, there exists an element $\hat{\theta}$ such that $\hat{\theta} \mapsto \mathrm{Id}$ which means $\theta\hat{\theta} = \mathrm{Id}$. Thus, $\hat{\theta}$ is right-inverse to θ. From associativity we obtain

$$\theta' = \theta'(\theta\hat{\theta}) = (\theta'\theta)\hat{\theta} = \hat{\theta}.$$

The proof for θ' being right-inverse is analogous. □

Lemma 8.32. *An element $u \in \Theta$ satisfies $u \trianglelefteq \theta$ for all $\theta \in \Theta$ if and only if it is invertible.*

Proof. Assume $u \trianglelefteq \theta$ for all $\theta \in \Theta$. Especially $u \trianglelefteq \mathrm{Id}$ which means there is an element u' with $u'u = \mathrm{Id}$. Thus, u' is left-inverse to u which is invertible due to Lemma 8.31. On the other hand, assume there is an inverse u' to u. Then $(\theta u')u = \theta(u'u) = \theta$ for all $\theta \in \Theta$. Hence, $u \trianglelefteq \theta$. □

Corollary 8.33. *If $\trianglelefteq$ is a partial order, the identity Id is the unique least element in Θ.*

Proof. The identity Id is obviously invertible and therefore satisfies $\mathrm{Id} \trianglelefteq \theta$ for all $\theta \in \Theta$. Then assume $\theta \trianglelefteq \mathrm{Id}$ for some $\theta \in \Theta$. This implies $\theta = \mathrm{Id}$, since $\trianglelefteq$ is a partial order. Hence, Id is a least element. As least elements are unique in a partially ordered set, this completes the proof. □

Remark 8.34. Corollary 8.33 implies that the fundamental network of a feedforward network has the unique least element Id which receives an input from any other cell in the network. △

8.3 Equivalence

In the previous sections, we investigated different network structures and their relations. Combining Definition 8.1, Corollary 8.7, Lemma 8.9 and Proposition 8.19, we see that all of them describe the same structural features.

Theorem 8.35. *A homogeneous coupled cell network is a feedforward network if and only if it satisfies one of the following equivalent conditions:*

(i) There are no directed loops involving 2 or more nodes in the network.

(ii) The preorder $\trianglelefteq$ is a partial order.

(iii) The monoid of input maps Σ is L-trivial.

(iv) The fundamental network is a feedforward network.

Definition 8.36. *From now on, we call a network a* feedforward network *if it satisfies one of the equivalent conditions in Theorem 8.35.*

In the remainder of this text, we therefore switch freely between the definitions and the terms with which we refer to the networks.

Remark 8.37. As mentioned in the introduction, an often considered generalization of feedforward chains is that of so-called *layered feedforward networks*. In such a network the cells are partitioned in layers and the feedforward structure is only with respect to these layers, i.e. if we collapse each layer to a single node we obtain a feedforward chain. Most notably we would like to mention Soares [99]. Therein, layered feedforward networks, their quotients and lifts as well as the lifting bifurcation problem are investigated thoroughly. It is shown that these networks exhibit similar steady state bifurcations as the ones presented in Chapter 10. As a matter of fact, completing the set of input maps for the networks considered in [99], we obtain networks that are included in our framework of feedforward networks. However, no self-loops (except for the internal dynamics governed by $\mathrm{Id} \in \Sigma$ and for the maximal cells) are possible. As a result, the branching patterns in the more general case considered in Chapter 10 are more complex. Note that in our more general definition of feed-forward networks, we may also group nodes together in layers, where we allow for self-loops and arrows that skip layers. In fact, it can be shown that we may give a definition in terms of layers that is equivalent to the definitions in Theorem 8.35. However, we decided not to include the precise definition here, as we deemed the 'graphical' and 'order theoretic' definitions to be more natural. $\triangle$

Chapter 9

Algebraic Structure of the Regular Representation

In this chapter we use the equivalent characterizations of feedforward structure provided in Theorem 8.35 in order to investigate spectral properties of network maps. Furthermore, we gather insight into the symmetries of the fundamental network. In particular, the additional structure of the network allows us to characterize the decomposition of the regular representation into indecomposable components. Recall that this solves the issue of the crucial *Step* 2 in the classification of all generic bifurcations of steady states as described in Section 6.1 for feedforward networks. Theorem 5.11, Theorem 5.12, and Remark 5.13 allow to deduce all possible generalized kernels and center subspaces in a generic bifurcation of steady states of a fundamental network from the indecomposable components of the right regular representation. This chapter is Section 3 of our preprint [84].

From now on, we restrict to the case $V = \mathbb{R}$ so that the total phase spaces are $\bigoplus_{p \in C} \mathbb{R} \cong \mathbb{R}^N$ and $\bigoplus_{\sigma \in \Sigma} \mathbb{R} \cong \mathbb{R}^n$ for the original and the fundamental network respectively. The first result shows why we are interested in the loop-type relation: there is a one-to-one correspondence between loop-types and eigenvalues of a linear admissible map.

Theorem 9.1. *In a feedforward network, the number of different loop-types of the nodes (that is, the number of equivalence classes under $\multimap$) equals the maximal number of different eigenvalues a linear admissible map can have.*

Proof. Recall from Lemma 8.13 that a linear admissible map for a feedforward network is upper triangular. Hence, the maximal number of eigenvalues is just the maximal number of distinct values the diagonal entries can attain. Similar to Section 4.2, we define the standard basis of the total phase space using the Kronecker delta $\delta_{p,q}$ – which equals 1 if $p = q$ and 0 otherwise – as $\{Y^q\}_{q \in C}$ given by $(Y^q_p)_{p \in C} = (\delta_{p,q})_{p \in C} \in \bigoplus_{p \in C} \mathbb{R}$. Note that this basis respects the labeling of cells chosen as in Lemma 8.13 or Remark 8.20 so that the linear admissible maps with respect to this basis are upper triangular. Recall furthermore from Corollary 3.9 that the algebra of linear admissible maps is spanned by the adjacency matrices B_σ for $\sigma \in \Sigma$, defined by $(B_\sigma(x))_p = x_{\sigma(p)}$ for $p \in C$. That is, any linear admissible map is of the form

$$\sum_{\sigma \in \Sigma} b_\sigma B_\sigma,$$

with $b_\sigma \in \mathbb{R}$ for all $\sigma \in \Sigma$. For a node $p \in C$, the (p,p)-entry of this matrix is given by

$$\left[\sum_{\sigma \in \Sigma} b_\sigma B_\sigma(Y^p)\right]_p = \sum_{\sigma \in \Sigma} b_\sigma Y^p_{\sigma(p)} = \sum_{\sigma \in \Sigma} b_\sigma \delta_{p,\sigma(p)} = \sum_{\sigma(p)=p} b_\sigma \,.$$

Hence, for two nodes $p, q \in C$ the (p,p)-entry and the (q,q)-entry are always the same – i.e. for any choice of coefficients $b_\sigma \in \mathbb{R}$ –, if and only if $p \multimap q$. This shows that the number of different eigenvalues is at most the number of loop-types. If p and q do not have the same loop-type, then for a dense open set of values $(b_\sigma)_{\sigma\in\Sigma}$, the p-th and q-th diagonal entries of $\sum_{\sigma\in\Sigma} b_\sigma B_\sigma$ are distinct. Intersecting these sets for all pairs of nodes with a different loop-type, we find a dense open set of values $(b_\sigma)_{\sigma\in\Sigma}$ for which $\sum_{\sigma\in\Sigma} b_\sigma B_\sigma$ has as many different eigenvalues as there are loop-types. This proves the theorem. □

In the remainder of this chapter we focus on fundamental networks. For these, the upper triangular structure of linear admissible maps and in particular Theorem 9.1 provide insight into the decomposition of the total phase space into indecomposable components. As in Remark 3.29, a general linear fundamental network map is of the form $L = \sum_{\sigma\in\Sigma} b_\sigma B_\sigma \in \mathrm{End}_\Sigma\left(\bigoplus_{\sigma\in\Sigma}\mathbb{R}\right)$ with $b_\sigma \in \mathbb{R}$ for all $\sigma \in \Sigma$. Here B_σ is given by $(B_\sigma X)_\tau = X_{\sigma\tau}$ for $\sigma,\tau \in \Sigma$ and $X \in \bigoplus_{\sigma\in\Sigma}\mathbb{R}$. This characterization will frequently be used without explicit mention.

Theorem 9.2. *The regular representation of an L-trivial monoid decomposes into all non-isomorphic indecomposable representations of real type. The number of indecomposable subrepresentations in a decomposition equals the number of different loop-types in the fundamental network.*

Proof. As the linear admissible maps for the fundamental network are all upper triangular, they only have real eigenvalues. Recall from Proposition 4.8 that an indecomposable component of complex or quaternionic type, as well as multiple isomorphic indecomposable components, would imply some admissible maps have imaginary eigenvalues, as the linear admissible maps are precisely the endomorphisms of the right regular representation $\mathrm{End}_\Sigma\left(\bigoplus_{\sigma\in\Sigma}\mathbb{R}\right)$. Therefore, all components are necessarily non-isomorphic and of real type.

The second part of the theorem follows almost immediately from the first. As all indecomposable subrepresentations are pairwise non-isomorphic and of real type, the number of components equals the maximal number of eigenvalues a linear admissible map for the fundamental network can have. By Theorem 9.1 this number equals the number of different loop-types in the fundamental network. □

Remark 9.3. One can easily determine the number of indecomposable representations from the multiplication table of an L-trivial monoid. To that end label the columns by elements of the monoid and relate them by stating that columns σ and τ are related if the places where the element σ appears in column σ are the same as the places where the element τ appears in column τ. The size of a maximal set of unrelated columns is then the number of indecomposable representations in a decomposition of the regular representation. △

The remainder of this chapter is devoted to deepening our understanding of the decomposition of the regular representation or, more notably, of possible center subspaces of linear network maps. Using the considerations from before – in particular the upper triangularity of the endomorphisms – we introduce a procedure to compute equivariant projections onto each component of the decomposition into indecomposable subrepresentations. These projections fully determine the decomposition of the regular representation (compare to Section 4.2). A general linear fundamental network map is of the form $L = \sum_{\sigma\in\Sigma} b_\sigma B_\sigma$. If Σ is an L-trivial monoid, its eigenvalues are given by the diagonal elements

$$\lambda_\sigma(L) = \sum_{\tau\sigma=\sigma} b_\tau \tag{9.1}$$

(note that $\lambda_\sigma(B_\tau) = (B_\tau)_{\sigma,\sigma} = 1$ if $\tau\sigma = \sigma$ and 0 otherwise). We see that $\lambda_\sigma(L) = \lambda_\tau(L)$ for all $L \in \mathrm{End}_\Sigma\left(\bigoplus_{\sigma\in\Sigma}\mathbb{R}\right)$ if and only if $\sigma \in \Sigma$ and $\tau \in \Sigma$ have the same loop-type: $\sigma \multimap\!\!\circ\, \tau$ (see the proof of Theorem 9.1). After choosing a suitable ordering of the elements of Σ, we know that L may be written as

$$L = \begin{pmatrix} \lambda_{\sigma_1}(L) & \bullet & \dots & \bullet \\ 0 & \lambda_{\sigma_2}(L) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bullet \\ 0 & \dots & 0 & \lambda_{\sigma_n}(L) \end{pmatrix}. \tag{9.2}$$

Next, we choose a decomposition of $\bigoplus_{\sigma\in\Sigma}\mathbb{R}$ into indecomposable representations:

$$\bigoplus_{\sigma\in\Sigma}\mathbb{R} = W_1 \oplus \dots \oplus W_m. \tag{9.3}$$

As stated before, from the fact that the eigenvalues of any admissible map L are all real, we conclude that $W_1,\dots,W_m$ are all of real type and mutually non-isomorphic (see Theorem 9.2). Furthermore, we have seen that m equals the maximal number of different eigenvalues any admissible map can have. There are values of $(b_\sigma)_{\sigma\in\Sigma}$ for which the eigenvalues $\lambda_\sigma(L) = \sum_{\tau\sigma=\sigma} b_\tau$ take different values for monoid elements σ with different loop-type. Hence, m also equals the number of different loop-types, that is, the number of equivalence classes under the loop-type relation. For later use let us denote by $\tau_1,\tau_2,\dots,\tau_m \in \Sigma$ a full set of representatives for the loop-type relation. Note that the standard basis – which corresponds to the cells of the network – does not respect the decomposition on the right hand side of Equation (9.3). Nevertheless, there is a basis of $\bigoplus_{\sigma\in\Sigma}\mathbb{R}$ respecting the decomposition and an equivariant isomorphism $\psi\colon \bigoplus_{\sigma\in\Sigma}\mathbb{R} \to \bigoplus_{\sigma\in\Sigma}\mathbb{R}$ that exchanges the bases. In particular, with respect to the decomposition on the right hand side of (9.3) we may write

$$\Psi(L) = \psi\circ L\circ\psi^{-1} = \begin{pmatrix} c_1\mathbb{1}_{W_1}+N_1 & L^{1,2} & \dots & L^{1,m} \\ L^{2,1} & c_2\mathbb{1}_{W_2}+N_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & L^{m-1,m} \\ L^{m,1} & \dots & L^{m,m-1} & c_m\mathbb{1}_{W_m}+N_m \end{pmatrix} \tag{9.4}$$

for some real numbers $c_1,\dots,c_m$. Here N_i is an equivariant nilpotent map from W_i to itself ($N_i \in \mathrm{End}_\Sigma^{\mathrm{nil}}(W_i)$) and every $L^{i,j}$ ($i\neq j$) is an equivariant map between non-isomorphic indecomposable representations ($L^{i,j} \in \mathrm{Hom}_\Sigma(W_j,W_i)$). Note that the form of the matrix in (9.4) relies crucially on the fact that the W_i are mutually non-isomorphic and of real type.

Lemma 9.4. *The numbers c_i in* (9.4) *are the eigenvalues of L, i.e. we have $c_i = c_i(L) = \lambda_{\tau_i}(L)$ for all $i\in\{1,\dots,m\}$, possibly after a reordering of the indices.*

Proof. The proof relies mainly on techniques presented in Chapter 5 and their generalizations in Nijholt and Rink [77] – in particular, the result follows from Proposition 4.5 in Nijholt and Rink [77] and the representation of endomorphisms of an indecomposable representation of real type in (4.4). Proposition 4.5 in [77] makes use of the well-known fact that two quadratic matrices A,B of the same size with $\mathrm{tr}\left(A^k\right) = \mathrm{tr}\left(B^k\right)$ for all $k\in\mathbb{N}$ have the same eigenvalues with the same algebraic multiplicities (compare to the proof of Proposition 4.8) which is also crucial here. From the fact that $\mathrm{End}_\Sigma^{\mathrm{nil}}(W_i) \subset \mathrm{End}_\Sigma(W_i)$ is an ideal, it follows that the invertible parts of the diagonal entries in the representation (9.4) are multiplicative when taking powers. More precisely

$$\left(\Psi(L)^2\right)_{j,j} = c_j^2\mathbb{1}_{W_j} + \left[2c_jN_j + \sum_{i\neq j} L^{j,i}L^{i,j}\right],$$

where $j = 1,\dots,m$. Due to the fact that the nilpotent endomorphisms form an ideal and due to Proposition 4.9 all terms in the parenthesis are nilpotent and so is their sum. This again relies on the fact that the W_i are pairwise non-isomorphic. Inductively, we obtain

$$\left(\Psi(L)^k\right)_{j,j} = c_j^k\mathbb{1}_{W_j} + \tilde{N}_j^k$$

where $k \in \mathbb{N}$ and $\tilde{N}_j^k \in \mathrm{End}_\Sigma^{\mathrm{nil}}(W_j)$. Writing

$$\Psi(L) = D + N = \begin{pmatrix} c_1 \mathbb{1}_{W_1} & 0 & \dots & 0 \\ 0 & c_2 \mathbb{1}_{W_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & c_m \mathbb{1}_{W_m} \end{pmatrix} + \begin{pmatrix} N_1 & L^{1,2} & \dots & L^{1,m} \\ L^{2,1} & N_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & L^{m-1,m} \\ L^{m,1} & \dots & L^{m,m-1} & N_m \end{pmatrix},$$

where $0 \in \mathrm{Hom}_\Sigma(W_j, W_i)$ is the trivial equivariant map, we obtain

$$\mathrm{tr}\left(\Psi(L)^k\right) = \mathrm{tr}\left(D^k\right)$$

for all $k \in \mathbb{N}$, since addition of a nilpotent operator does not change the trace. Therefore, the eigenvalues and their algebraic multiplicity of $\Psi(L)$ and those of D coincide. The eigenvalues of D, however, are precisely the entries $c_1, \dots, c_m$. As L has the same eigenvalues with equal algebraic multiplicity as $\Psi(L)$, this completes the proof. $\square$

The following lemma shows that the reordering in the previous lemma may be done for all L simultaneously.

Lemma 9.5. *There exists a reordering of the W_i in equation* (9.3) *so that we have $c_i = c_i(L) = \lambda_{\tau_i}(L)$ for all $i \in \{1, \dots, m\}$ and for all linear admissible maps L (simultaneously). In particular, the number of $\sigma \in \Sigma$ that are loop-type equivalent to τ_i (i.e. the number of times $\lambda_{\tau_i}(L)$ appears on the diagonal) equals $\dim W_i$.*

Proof. We prove this result by going back and forth between the two representations (9.2) and (9.4) of a single operator L. For $i \in \{1, \dots, m\}$ we denote by $P_i \in \mathrm{End}_\Sigma\left(\bigoplus_{\sigma \in \Sigma} \mathbb{R}\right)$ the map such that $\Psi(P_i)$ is the projection onto W_i along the decomposition (9.3). By the equality of the space of linear admissible maps and the space of endomorphisms of the regular representation, we see that the P_i are linear network maps. As P_i is a (non-zero) projection operator, it has only eigenvalues 0 and 1, with 1 appearing at least once. We therefore have $\lambda_{\tau_j}(P_i) = 1$ for some values of $j \in \{1, \dots, m\}$, say for $j \in \Omega_i \subset \{1, \dots, m\}$. In particular then, we have $\lambda_{\tau_j}(P_i) = 0$ for $j \in \{1, \dots, m\} \setminus \Omega_i$.

Next, it follows from the upper triangular shape that $\lambda_\sigma(LL') = \lambda_\sigma(L)\lambda_\sigma(L')$ for all admissible maps L and L' and for all $\sigma \in \Sigma$. In particular, we see that $\lambda_{\tau_l}(P_i)\lambda_{\tau_l}(P_j) = \lambda_{\tau_l}(P_i P_j) = \lambda_{\tau_l}(0) = 0$ for all $i, j, l \in \{1, \dots, m\}$ with $i \neq j$. This means that $\lambda_{\tau_l}(P_i)$ and $\lambda_{\tau_l}(P_j)$ cannot be simultaneously equal to 1, and we conclude that $\Omega_i \cap \Omega_j = \emptyset$ whenever $i \neq j$. To summarize, we have m subsets $(\Omega_i)_{i=1}^m$ of $\{1, \dots, m\}$, each one is non-empty and they are all pairwise disjoint. Hence, the Ω_i are all one-point sets, and we may reorder the indices in equation (9.3) so that $\Omega_i = \{i\}$. Therefore, we may relate loop-type to indecomposable representations and P_i has diagonal entries 1 at the (σ, σ) position if σ is of the loop-type that corresponds to the subrepresentation W_i and 0 in all other diagonal entries.

Lastly, we use that LP_i and $\Psi(LP_i) = \Psi(L)\Psi(P_i)$ have the same eigenvalues. Because $\lambda_\sigma(LP_i) = \lambda_\sigma(L)\lambda_\sigma(P_i)$ for all $\sigma \in \Sigma$, we see that the eigenvalues of LP_i are given by $\{0, \lambda_{\tau_i}(L)\}$ (or simply $\{\lambda_{\tau_i}(L)\}$ if $m = 1$). On the other hand, as $\Psi(P_i)$ is the projection onto W_i, we see that the eigenvalues of $\Psi(L)\Psi(P_i)$ are given by $\{0, c_i(L)\}$ (or $\{c_i(L)\}$ for $k = 1$). Hence, we see that necessarily $\lambda_{\tau_i}(L) = c_i(L)$. As the $c_i(L)$ and the $\lambda_{\tau_i}(L)$ both denote the eigenvalues of L, choosing an operator L such that all the $c_i(L)$ are distinct proves that the number of $\sigma \in \Sigma$ that are loop-type equivalent to τ_i (so the number of times $\lambda_{\tau_i}(L)$ appears on the diagonal) equals $\dim W_i$ (see the proof of Lemma 9.4). $\square$

Recall that we fixed a set of representatives $\tau_1, \tau_2, \dots, \tau_m \in \Sigma$ for the loop-types, and suppose furthermore that the components in (9.3) are ordered as in Lemma 9.5. In other words, we may assume that $c_i = c_i(L) = \lambda_{\tau_i}(L)$ for all $i \in \{1, \dots, m\}$ and for all maps L. As the numbers $\lambda_{\tau_i}(L)$

denote the eigenvalues of L, and as the form (9.4) for $\Psi(L)$ implies there are no relations between the eigenvalues of L, we see that the $\lambda_{\tau_i}(L)$ are necessarily linearly independent. In particular, the equations $\lambda_{\tau_i}(L) = \delta_{i,j}$ (the Kronecker delta) for all $i \in \{1,\dots,m\}$ and for some given j have a solution (and often more than one). This observation motivates the following theorem.

Theorem 9.6. *Fix $j \in \{i,\dots,m\}$. Suppose the numbers $(b_\sigma)_{\sigma\in\Sigma}$ are such that*

$$\sum_{\tau\tau_i=\tau_i} b_\tau = \delta_{i,j} \tag{9.5}$$

for all $i \in \{1,\dots,m\}$. Setting $R_j = \sum_{\sigma\in\Sigma} b_\sigma B_\sigma$ we have that $Q_j = \mathbb{1} - (\mathbb{1} - R_j^n)^n$ is an equivariant projection with image isomorphic to W_j, where we abbreviate $\mathbb{1} = \mathbb{1}_{\bigoplus_{\sigma\in\Sigma}\mathbb{R}} \in \mathrm{End}_\Sigma\left(\bigoplus_{\sigma\in\Sigma}\mathbb{R}\right)$.

More generally, fix some subset $J \subset \{1,\dots,m\}$ and assume the numbers $(b_\sigma)_{\sigma\in\Sigma}$ are such that

$$\sum_{\tau\tau_i=\tau_i} b_\tau = \delta_{i\in J}, \tag{9.6}$$

where $\delta_{i\in J}$ is a generalization of the Kronecker delta with $\delta_{i\in J} = 1$ if $i \in J$ and 0 otherwise. Then setting $R_J = \sum_{\sigma\in\Sigma} b_\sigma B_\sigma$ makes $Q_J = \mathbb{1} - (\mathbb{1} - R_J^n)^n$ a projection onto $\bigoplus_{j\in J} W_j$.

We will need the following lemma.

Lemma 9.7. *Let L be a linear fundamental network map for an L-trivial monoid. If L has only zeroes and ones on its diagonal, then $C = \mathbb{1} - (\mathbb{1} - L^n)^n$ is a projection with the same diagonal as L.*

Proof. As L has only zeroes and ones on its diagonal, it follows that $L(\mathbb{1}-L)$ has vanishing diagonal. Therefore, we see that $L(\mathbb{1}-L)$ is nilpotent and we get

$$[L(\mathbb{1}-L)]^n = L^n(\mathbb{1}-L)^n = 0,$$

where $n = \#\Sigma = \dim\bigoplus_{\sigma\in\Sigma}\mathbb{R}$. Another fact that we will need is that

$$\mathbb{1} - L^n = (\mathbb{1}-L)(\mathbb{1} + L + L^2 + \cdots + L^{n-1}).$$

Setting $R(L) = (\mathbb{1} + L + L^2 + \cdots + L^{n-1})$ we get

$$\mathbb{1} - L^n = (\mathbb{1}-L)R(L) = R(L)(\mathbb{1}-L).$$

Replacing L with $(\mathbb{1}-L^n)$, we see that $\mathbb{1} - (\mathbb{1}-L^n)^n$ is divisible by L^n and we write

$$\mathbb{1} - (\mathbb{1}-L^n)^n = L^nS(L) = S(L)L^n$$

Using these representations and the commutativity relations, we calculate $C - C^2$:

$$\begin{aligned}
C - C^2 &= [\mathbb{1} - (\mathbb{1}-L^n)^n] - [\mathbb{1} - (\mathbb{1}-L^n)^n]^2 \\
&= [\mathbb{1} - (\mathbb{1}-L^n)^n][\mathbb{1} - (\mathbb{1} - (\mathbb{1}-L^n)^n)] \\
&= [\mathbb{1} - (\mathbb{1}-L^n)^n][(\mathbb{1}-L^n)^n] \\
&= S(L)\underbrace{L^n(\mathbb{1}-L)^n}_{=\mathbf{0}}R(L)^n \\
&= 0.
\end{aligned}$$

This shows that $C^2 = C$, so that $C = \mathbb{1} - (\mathbb{1}-L^n)^n$ is indeed a projection.

To show that L and C have the same diagonal, we write $L = D + N$ for D the diagonal of L and with N some strictly upper triangular map (i.e. N has a vanishing diagonal). As D consists of only zeroes and ones, we see that $D^2 = D$ and likewise $(\mathbb{1}-D)^2 = \mathbb{1} - D$. Moreover, any product involving N has again a vanishing diagonal. For this reason, we will write $N, N_1, \dots$ for any term with vanishing diagonal. Using this notation we get

$$\begin{aligned}
\mathbb{1} - (\mathbb{1} - L^n)^n &= \mathbb{1} - (\mathbb{1} - (D+N)^n)^n = \mathbb{1} - (\mathbb{1} - D^n + N_1)^n \\
&= \mathbb{1} - (\mathbb{1} - D + N_1)^n = \mathbb{1} - (\mathbb{1} - D)^n + N_2 \\
&= \mathbb{1} - (\mathbb{1} - D) + N_2 = D + N_2.
\end{aligned}$$

This shows that L and C indeed have the same diagonal D which finishes the proof. $\square$

Proof of Theorem 9.6. As the eigenvalues of R_j are exactly given by $\lambda_{\tau_i}(R_j) = \sum_{\tau\tau_i=\tau_i} b_\tau = \delta_{i,j}$ (see (9.1)), we see that R_j has only zeroes and ones on its diagonal. By Lemma 9.7, the map $Q_j = \mathbb{1} - (\mathbb{1} - R_j^n)^n$ is a projection with the same diagonal, so that $\lambda_{\tau_i}(Q_j) = \delta_{i,j}$. The map Q_j is furthermore equivariant by construction.

Next, we consider $\Psi(Q_j)$. By Lemma 9.5, $\Psi(Q_j)$ is of the form (9.4) with $c_i = \lambda_{\tau_i}(Q_j) = \delta_{i,j}$ (and with the N_i and $L^{i,j}$ not necessarily vanishing). As $\Psi(Q_j)$ is a projection, we know that the dimension of its image is given by the number of non-zero eigenvalues of $\Psi(Q_j)$. Apparently, this number equals $\dim W_j$, as only the c_j are non-zero. We conclude that

$$\dim \operatorname{im}(Q_j) = \dim W_j. \tag{9.7}$$

Next, we construct the equivariant map $\Xi_j = \Psi(Q_j) \circ I_j$ from W_j into the image of $\Psi(Q_j)$, where I_j denotes the inclusion of W_j into $\bigoplus_{\sigma\in\Sigma} \mathbb{R}$ in the decomposition of (9.3). Writing $\Psi(Q_j)$ as in (9.4) with $c_i = \delta_{i,j}$, we see that Ξ_j is given by

$$\Xi_j(Y) = (L^{j,1}Y, \dots, L^{j,j-1}Y, Y + N_jY, L^{j,j+1}Y, \dots, L^{j,k}Y)^T \tag{9.8}$$

for $Y \in W_j$. Solving the equation $\Xi_j(Y) = 0$ gives $Y + N_jY = 0$, or $N_jY = -Y$. As N_j is nilpotent, we conclude that Y has to vanish. Therefore Ξ_j is injective, and so $\psi^{-1}\Xi_j \colon W_j \to \operatorname{im}(Q_j)$ is injective as well. From (9.7) we conclude that indeed $W_j \cong \operatorname{im}(Q_j)$.

The generalization, constructing Q_J for $J \subset \{1, \dots, m\}$, works almost exactly the same. Once again $\Psi(Q_J)$ is of the form (9.4) with $c_j = 1$ for $j \in J$ and 0 otherwise. In particular $\dim \operatorname{im}(Q_J) = \dim \bigoplus_{j\in J} W_j$ as in (9.7). Let I_J be the inclusion of $\bigoplus_{j\in J} W_j$ into $\bigoplus_{\sigma\in\Sigma} \mathbb{R}$ in the decomposition of (9.3) and P_J the corresponding projection. Then the map

$$\Xi_J = P_J \circ \Psi(Q_J) \circ I_J \colon \bigoplus_{j\in J} W_j \to \bigoplus_{j\in J} W_j$$

is of the form

$$\Xi_J = \begin{pmatrix}
c_{j_1}\mathbb{1}_{W_{j_1}} + N_{j_1} & L^{j_1,j_2} & \dots & L^{j_1,j_k} \\
L^{j_2,j_1} & c_{j_2}\mathbb{1}_{W_{j_2}} + N_{j_2} & \ddots & \vdots \\
\vdots & \ddots & \ddots & L^{j_{k-1},j_k} \\
L^{j_k,j_1} & \dots & L^{j_k,j_{k-1}} & c_{j_k}\mathbb{1}_{W_{j_k}} + N_{j_k}
\end{pmatrix}$$

for the labeling $J = \{j_1, \dots, j_k\}$ (compare to (9.4)). Once again, the eigenvalues are given by the values c_{j_i} for all $i \in \{1, \dots, k\}$. As $j_i \in J$, these eigenvalues are all equal to 1 so that Ξ_J and $\psi^{-1}\Xi_J \colon \bigoplus_{j\in J} W_j \to \operatorname{im}(Q_J)$ are both injective (more precisely even bijective). As before we conclude $\bigoplus_{j\in J} W_j \cong \operatorname{im}(Q_J)$. $\square$

Constructing projections In order to obtain a projection onto the indecomposable component W_j from Theorem 9.6, we need to solve the equations (9.5)

$$\sum_{\tau\tau_i=\tau_i} b_\tau = \delta_{i,j}$$

simultaneously for all $i \in \{1, \ldots, m\}$. This step, however, can be performed implicitly by directly constructing linear maps with the 'correct' diagonal following an easy algorithm. To this end, we define $\Lambda_i = \{\tau \mid \tau\tau_i = \tau_i\} = \mathcal{L}_{\tau_i}$ as the set of those elements appearing in the sum in (9.5). Equivalently, Λ_i denotes those types of arrows that form a loop from a node to itself, for any node with the same loop-type as τ_i. Note that all Λ_i contain the identity element Id.

For any $\sigma \in \Sigma$ not equal to Id, we define the operator

$$D_\sigma = B_{\mathrm{Id}} - B_\sigma \,. \tag{9.9}$$

Note that by construction we have

$$\lambda_{\tau_i}(D_\sigma) = \begin{cases} 1 & \text{if} \quad \sigma \notin \Lambda_i \\ 0 & \text{if} \quad \sigma \in \Lambda_i \end{cases} \tag{9.10}$$

as in (9.1). Likewise, we have

$$\lambda_{\tau_i}(B_\sigma) = \begin{cases} 0 & \text{if} \quad \sigma \notin \Lambda_i \\ 1 & \text{if} \quad \sigma \in \Lambda_i. \end{cases} \tag{9.11}$$

Therefore, we may simply set

$$R_j = \prod_{\sigma \in \Lambda_j} B_\sigma \prod_{\tau \notin \Lambda_j} D_\tau \,, \tag{9.12}$$

where the product is taken in any order. It follows that

$$\lambda_{\tau_i}(R_j) = \prod_{\sigma \in \Lambda_j} \lambda_{\tau_i}(B_\sigma) \prod_{\tau \notin \Lambda_j} \lambda_{\tau_i}(D_\tau) \,, \tag{9.13}$$

which equals 1 precisely when any σ in Λ_j is also in Λ_i and when any σ not in Λ_j is also not in Λ_i. In other words, precisely when $\Lambda_i = \Lambda_j$, and so when $i = j$. This shows that the 'pre-projections' R_j can indeed be constructed explicitly.

For the generalization to a projection onto $\bigoplus_{j \in J} W_j$ for an arbitrary subset $J \subset \{1, \ldots, m\}$ note that solving (9.6) simultaneously for all $i \in \{1, \ldots, m\}$ is equivalent to solving the equations (9.5) for all $j \in J$ simultaneously for all $i \in \{1, \ldots, m\}$. Hence, we may define

$$R_J = \sum_{j \in J} R_j$$

where the R_j are defined as above.

Remark 9.8. The dual notion to L-triviality is called R-triviality. A monoid Θ is *R-trivial* if the sets $\theta\Theta$ are different for all $\theta \in \Theta$. In other words, if for $\theta, \theta' \in \Theta$ we have $\theta\Theta = \theta'\Theta$ then $\theta = \theta'$. Using classification results from SAITO [94], one can see that the input monoid Σ of a network in our formalism – not necessarily a fundamental network – is R-trivial if and only if for every two input maps $\sigma, \tau \in \Sigma$ the connected components of the network with only input maps σ and τ are the same as the connected components in the network with only input map $\sigma\tau$. In contrast to the feedforward structure for L-trivial monoids, there is, to our knowledge, no easily accessible way of describing networks with R-trivial monoids of input maps. However, a sufficient condition is that all the maps in the monoid have a different image, because if $\sigma\Sigma \subset \tau\Sigma$ then $\sigma = \tau\kappa$ for some $\kappa \in \Sigma$ so $\mathrm{im}\,(\sigma) \subset \mathrm{im}\,(\tau)$.

The duality of notions, however, allows us to compute algebraic properties in a similar way. As a matter of fact, the opposite monoid Σ^{op}– with reversed order of multiplication – of an R-trivial one is L-trivial. Hence, we may compute projections onto indecomposable subrepresentations in a similar way. The sets Λ_i need to be determined from Σ^{op}. The rest of the construction remains the same. △

Remark 9.9. The problem of decomposing regular representations of monoids – or their monoid algebras as in Section 4.2 – is open and actively studied in algebraic representation theory (an overview can be found in STEINBERG [105]). It turns out, that the algorithm for determining projections onto indecomposable components presented here can significantly be generalized. As a matter of fact, for an L-trivial (or R-trivial in the spirit of the previous remark) monoid a similar decomposition can be derived for regular modules of monoid algebras over any ring for which we know a complete set of primitive orthogonal idempotents (in particular over any field). This has led to the preprint NIJHOLT, RINK, and SCHWENKER [82] inspired by a similar result in BERG et al. [18]. We decided not to include the details here, as full generality (at least as far as we are aware of) in the algebraic setting is beyond the scope of this thesis. △

Chapter 10

Amplified Steady State Bifurcations in Feedforward Networks

In this chapter, we classify generic 1-parameter steady state bifurcations in feedforward networks. That is, we consider a network with cells $C = \{p_1, \dots, p_N\}$ and input maps $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ that satisfies the equivalent definitions in Theorem 8.35. It turns out, that instead of going through the process of Section 6.1 this classification can effectively be done by explicitly solving the bifurcation equations for a generic response function without the application of any reduction methods. In particular, the partial order $\trianglelefteq$ on C induces a partial order of the equations to be solved. This procedure has the advantage that we compute branching solutions in the original coordinates. In particular, the distinction of the cells is respected so that we can precisely see how the state of each individual cell varies in the bifurcation depending on some Taylor coefficients of the response function. We perform these computations in the case of one-dimensional internal dynamics. The methods introduced in Chapter 7 then show that steady state bifurcations for arbitrary finite-dimensional internal dynamics exhibit qualitatively the same branching patterns in the sense that the state of every cell branches with the same asymptotics. The explicit dependence on Taylor coefficients can obviously not be recovered from the one-dimensional internal dynamics case. These investigations will be made in Chapter 11. This chapter is Section 4 in our preprint [84].

We begin by recalling the setting of a bifurcation problem. Assuming the labeling of nodes is according to Remark 8.20, the parameter-dependent dynamics on the total phase space $\bigoplus_{p \in C} V \cong V^N$ is governed by

$$\dot{x} = \gamma_f(x, \lambda) = \begin{pmatrix} f(x_{\sigma_1(p_1)}, \dots, x_{\sigma_n(p_1)}, \lambda) \\ f(x_{\sigma_1(p_2)}, \dots, x_{\sigma_n(p_2)}, \lambda) \\ \vdots \\ f(x_{\sigma_1(p_N)}, \dots, x_{\sigma_n(p_N)}, \lambda) \end{pmatrix}, \tag{10.1}$$

where $\lambda \in \mathbb{R}$ and $\gamma_f(x, \lambda)_p$ depends only on those $q \in C$ with $q \trianglerighteq p$.

We aim at investigating generic bifurcations from a fully synchronous steady state. Without loss of generality, we assume this to be the origin and the bifurcation to occur for $\lambda = 0$. Hence, we assume

$$\gamma_f(0, 0) = 0,$$

which implies $f(0, 0) = 0$.

As before, a steady state bifurcation can only occur if the linearization $D_x \gamma_f(0, 0)$ is non-invertible. As the inputs of f are labeled by the elements $\sigma \in \Sigma$, we may define

$$a_\sigma = \partial_\sigma f(0, 0),$$

which is an arbitrary linear map on V, i.e. $a_\sigma \in \mathfrak{gl}(V)$. Furthermore, recall the definition

$$\mathcal{L}_p = \{\sigma \in \Sigma \mid \sigma(p) = p\}$$

as in Definition 8.16. Then p and q are of the same loop-type, $p \multimapboth q$, if and only if $\mathcal{L}_p = \mathcal{L}_q$. We compute the linearization to be the block-triangular matrix

$$D_x\gamma_f(0,0) = \begin{pmatrix} \sum_{\sigma\in\mathcal{L}_{p_1}} a_\sigma & \bullet & \cdots & \bullet \\ 0 & \sum_{\sigma\in\mathcal{L}_{p_2}} a_\sigma & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bullet \\ 0 & \cdots & 0 & \sum_{\sigma\in\mathcal{L}_{p_N}} a_\sigma \end{pmatrix}. \tag{10.2}$$

Hence, $D_x\gamma_f(0,0)$ is non-invertible, if and only if there is a node $\overline{p} \in C$ such that $0 \in \operatorname{spec}(\sum_{\sigma\in\mathcal{L}_{\overline{p}}} a_\sigma)$. Note that $\sum_{\sigma\in\mathcal{L}_p} a_\sigma = \sum_{\sigma\in\mathcal{L}_q} a_\sigma$, if $p \multimapboth q$. Thus, $0 \in \operatorname{spec}(\sum_{\sigma\in\mathcal{L}_p} a_\sigma)$ for all $p \multimapboth \overline{p}$. We call these nodes *critical*. Furthermore, as $a_\sigma \in \mathfrak{gl}(V)$ arbitrary, generically $\sum_{\sigma\in\mathcal{L}_p} a_\sigma$ does not have an eigenvalue 0 if p is not of the same loop-type as $\overline{p}$. Hence, we call the loop-type of $\overline{p}$ *critical* and all other loop-types *non-critical*.

Remark 10.1. In the tensor notation from Proposition 7.2 this linear map has the form

$$D_x\gamma_f(0,0) = \sum_{\sigma\in\Sigma} B_\sigma \otimes a_\sigma,$$

Here the maps B_σ are the upper triangular adjacency matrices in the case $V = \mathbb{R}$. However, as we are only interested in the diagonal elements, the non-tensor notation is more convenient. △

The steady state bifurcation problem is to find solutions to

$$\gamma_f(x,\lambda) = 0 \tag{10.3}$$

locally around $(0,0) \in \bigoplus_{p\in C} V \times \mathbb{R}$ for a generic f satisfying the bifurcation assumptions (B)

(B.i) $f(0,0) = 0$;

(B.ii) there exists $\overline{p} \in C$ such that $0 \in \operatorname{spec}(\sum_{\sigma\in\mathcal{L}_{\overline{p}}} a_\sigma)$.

Lemma 10.2. *Maximal cells are either all critical or all non-critical.*

Proof. This follows from the fact that every maximal cell p is fixed by the entire monoid, i.e. $\mathcal{L}_p = \Sigma$ – compare to Lemma 8.21 and Corollary 8.22. □

Hence, when the maximal cells are critical all non-maximal cells may be assumed not to be and vice versa when a non-maximal cell is critical all maximal cells may be assumed not to be.

We explicitly compute the generic steady state bifurcation behavior in individual cells in the case of one-dimensional internal dynamics, that is $V = \mathbb{R}$. The results and their proofs are rather notation heavy. However, the branching solutions can be summarized informally as follows:

(i) Maximal cells evolve asymptotically as $\sim \lambda$ if they are non-critical and as $\sim \pm\sqrt{\lambda}$ otherwise.

(ii) Non-critical cells are, up to lowest order, linear in their inputs. Hence up to lowest order they have the same asymptotics as the lowest possible order of their inputs.

(iii) Critical cells are, up to lowest order, the square root of their lowest order inputs. We refer to this phenomenon as *amplification*.

These results follow from an inductive investigation of the bifurcations in individual cells with respect to $\trianglelefteq$. We have to carefully distinguish when a bifurcation occurs for positive or negative values of λ. Furthermore, taking square roots of inputs is only possible if the signs of inputs are suitable. This results in restrictions on system parameters, i.e. Taylor coefficients or partial derivatives of the governing function f.

The key part in the computations in this chapter is the fact that for an arbitrary cell $p \in C$ the function $\gamma_f(x,\lambda)_p = f(x_{\sigma_1(p)},\dots,x_{\sigma_n(p)},\lambda)$ depends only on those $q \in C$ with $q \trianglerighteq p$. Together with the bifurcation assumptions (B) this allows us to Taylor expand the governing function as

$$\begin{aligned}
0 = f(x_{\sigma_1(p)},\dots,x_{\sigma_n(p)},\lambda) = &\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}x_p^2 + \sum_{\sigma\in\mathcal{L}_p} a_\sigma x_p + \sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda}\lambda x_p \\
&+ \sum_{\substack{p\triangleleft q=\tau(p):\\ \tau\notin\mathcal{L}_p}} a_\tau x_q + \ell\lambda + \sum_{\substack{p\triangleleft q=\tau(p):\\ \tau\notin\mathcal{L}_p}} f_{\tau\lambda}\lambda x_q \\
&+ 2\sum_{\substack{\sigma\in\mathcal{L}_p\\ \tau\notin\mathcal{L}_p:\\ p\triangleleft q=\tau(p)}} f_{\sigma\tau}x_p x_q + \sum_{\substack{\sigma,\tau\notin\mathcal{L}_p\\ p\triangleleft q,s=\sigma(p),\tau(p)}} f_{\sigma\tau}x_q x_s + f_{\lambda\lambda}\lambda^2 \\
&+ \mathcal{O}\left(\|((x_q \mid q \trianglerighteq p),\lambda)\|^3\right).
\end{aligned} \tag{10.4}$$

The constants $f_{\sigma\tau}, f_{\tau\lambda}, \ell$, and $f_{\lambda\lambda}$ are defined to be partial derivatives of f in directions labeled by the input functions $\sigma \in \Sigma$, similar to the a_σ before, or by the parameter λ:

$$f_{\sigma\tau} = \frac{1}{2}\partial_{\sigma\tau}f(0,0) \quad \text{(hence } f_{\sigma\tau} = f_{\tau\sigma}\text{)}, \qquad \ell = \partial_\lambda f(0,0),$$
$$f_{\tau\lambda} = \partial_{\tau\lambda}f(0,0) = \partial_{\lambda\tau}f(0,0), \qquad f_{\lambda\lambda} = \frac{1}{2}\partial_{\lambda\lambda}f(0,0).$$

Note that in the case $V = \mathbb{R}$ all these constants are real numbers and especially (B.ii) becomes $\sum_{\sigma\in\mathcal{L}_{\overline{p}}} a_\sigma = 0$ for a critical cell $\overline{p}$.

It turns out that inductively solving (10.4) results in significantly different solutions depending on the two possible cases presented in Lemma 10.2 – either the maximal cells are critical or not. Even though the computations follow the general idea outlined above in both cases, it is convenient to separate the investigations.

Remark 10.3. In Chapter 9 we introduce a method to determine all possible (generic) center subspaces for the bifurcation problem in the fundamental network. However, it turns out that the form of the equations (10.4) together with the partial order $\trianglelefteq$ are sufficient to determine generic steady state bifurcations for a (not necessarily fundamental) feedforward network with one-dimensional internal dynamics. This has the additional advantage that the bifurcation behavior of each individual cell can be determined as outlined above. The hidden symmetries from the fundamental network in turn explain much of the underlying mechanisms, even if we do not use them explicitly here. Moreover, more complicated bifurcation problems may be investigated using the hidden symmetry formalism, such as pertaining to Hopf bifurcations, connecting orbits, multiple bifurcation parameters and higher dimensional internal dynamics. △

10.1 Maximal cells are critical

We start with the simpler case by assuming the maximal cells to be critical. From Lemma 10.2 we know that this is equivalent to $\sum_{\sigma\in\Sigma} a_\sigma = 0$ and to all non-maximal cells not being critical generically. This greatly simplifies the computations as there are fewer cases to take care of. Furthermore, the bifurcation condition (B.ii) is equivalent to

$$\sum_{\sigma\in\mathcal{L}_p} a_\sigma = -\sum_{\tau\notin\mathcal{L}_p} a_\tau \tag{10.5}$$

for every cell $p \in C$, which is a useful identity. We start by computing the bifurcation behavior of (10.4) for maximal cells, where we detect a saddle node bifurcation. Then we proceed inductively with respect to $\trianglelefteq$ to all non-maximal cells, whose state variables mimic the bifurcation behavior of their inputs.

Lemma 10.4. *Let $p \in C$ be maximal. The state variable x_p generically bifurcates as in one of the following two cases:*

(i)

$$x_p(\lambda) = D_p^{\pm} \cdot \sqrt{\lambda} + \mathcal{O}(|\lambda|)$$

for $\lambda > 0$ small, if

$$\frac{\ell}{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}} < 0.$$

Here

$$D_p^{\pm} = \pm\sqrt{-\frac{\ell}{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}}} \neq 0.$$

(ii)

$$x_p(\lambda) = D_p^{\pm} \cdot \sqrt{-\lambda} + \mathcal{O}(|\lambda|)$$

for $\lambda < 0$ small, if

$$\frac{\ell}{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}} > 0.$$

Here

$$D_p^{\pm} = \pm\sqrt{\frac{\ell}{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}}} \neq 0.$$

The bifurcating solutions are the same for all maximal cells.

Proof. As a maximal cell p only receives inputs from itself (see Lemma 8.21), the equations to be solved (10.4) only depend on the state variable of that specific cell. Hence, the bifurcation behavior is exactly the same for all maximal cells. As $\sum_{\sigma\in\Sigma} a_\sigma = 0$ it becomes

$$0 = \sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau} x_p^2 + \ell\lambda + \mathcal{O}(|\lambda|^2 + |x_p|^3 + |x_p||\lambda|). \tag{10.6}$$

In both cases of the lemma we employ the standard method to detect saddle node bifurcations (see for example MURDOCK [75]). Assume $\ell / \sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau} < 0$. We introduce a new variable $x_p = \mu y$ where $\mu = \sqrt{\lambda}$ for small $\lambda > 0$, hence $\mu > 0$. Equation (10.6) transforms into

$$0 = \sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau} \mu^2 y^2 + \ell\mu^2 + \mathcal{O}(|\mu|^4 + |\mu|^3|y|).$$

As $\mu > 0$, we may divide by μ^2 and obtain

$$0 = \sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau} y^2 + \ell + \mathcal{O}(|\mu|^2 + |\mu||y|) = g(y,\mu).$$

For $\mu = 0$ this is equation is solved by

$$\overline{y}^{\pm} = \pm\sqrt{-\frac{\ell}{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}}}.$$

Furthermore, $\frac{\partial}{\partial y} g(\overline{y}^{\pm}, 0) = 2\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}\overline{y}^{\pm}$, which, generically, does not equal 0. Hence, by the implicit function theorem, we obtain two branches of solutions

$$\overline{Y}^{\pm}(\mu) = \overline{y}^{\pm} + \mathcal{O}(|\mu|).$$

Transforming back into the original variables we obtain two branches

$$x_p(\lambda) = \pm\sqrt{-\frac{\ell}{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}}} \cdot \sqrt{\lambda} + \mathcal{O}(|\lambda|)$$

for small $\lambda > 0$, which completes the proof for the first case.

The case $\ell/\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau} < 0$ is very similar. We introduce $x_p = \mu y$ with $\mu = \sqrt{-\lambda}$ for $\lambda < 0$ with small absolute value. The equation to be solved becomes

$$0 = \sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau} y^2 - \ell + \mathcal{O}(|\mu|^2 + |\mu||y|) = g(y, \mu),$$

where we have divided by μ^2 again. For $\mu = 0$ this has two solutions

$$\overline{y}^{\pm} = \pm\sqrt{\frac{\ell}{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}}}$$

and $\frac{\partial}{\partial y} g(\overline{y}^{\pm}, 0) = 2\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}\overline{y}^{\pm} \neq 0$ generically. Once more we obtain two branches of solutions

$$\overline{Y}^{\pm}(\mu) = \overline{y}^{\pm} + \mathcal{O}(|\mu|)$$

by the implicit function theorem. Transforming back into the original coordinates, we obtain

$$x_p(\lambda) = \pm\sqrt{\frac{\ell}{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}}} \cdot \sqrt{-\lambda} + \mathcal{O}(|\lambda|)$$

for $\lambda < 0$ with small absolute value, which completes the proof. □

Remark 10.5. (i) The critical maximal cells simultaneously undergo a saddle node bifurcation, that is two steady state branches exist for either only positive or only negative values of λ. The sign of λ for which branching solutions occur is the same for all maximal cells. We refer to the former as the *supercritical* and to the latter as the *subcritical* case. Note that a generic response function f satisfying the bifurcation conditions (B) always fulfills one of the two assumptions from the previous lemma, as generically $\ell, \sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau} \neq 0$. However, we see that the generic bifurcation behavior is different in different regions of system parameter space, i.e. the space of partial derivatives of f.

(ii) Note, furthermore, that in both cases the equations for maximal cells are completely uncoupled. Hence, for a specific branch of solutions not all maximal cells need to evolve according to the same branch. In particular the choice of sign in $D_p^{\pm}$ may differ in different maximal cells. As a result, globally, when restricting only to maximal cells, we obtain 2^m branches of solutions where m is the number of maximal cells. △

For the non-maximal cells, we proceed inductively. We assume to know a specific branching pattern for all cells above a given cell p and compute the solutions for that cell. Hence, we need to distinguish between the super- and subcritical cases.

Lemma 10.6 (supercritical case). *Let p be non-maximal. Assume for all $q \rhd p$*

$$x_q(\lambda) = d_q \cdot \sqrt{\lambda} + \mathcal{O}(|\lambda|)$$

for small $\lambda > 0$ and some $d_q \in \mathbb{R} \setminus \{0\}$. Then

$$x_p(\lambda) = D_p \cdot \sqrt{\lambda} + \mathcal{O}(|\lambda|)$$

for small $\lambda > 0$, where

$$D_p = -\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau d_{\tau(p)}}{\sum_{\sigma\in\mathcal{L}_p} a_\sigma} = \frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau d_{\tau(p)}}{\sum_{\tau\notin\mathcal{L}_p} a_\tau} \neq 0.$$

Proof. As the maximal cells are critical and p is non-maximal, p is non-critical. Using the assumption on all $q \rhd p$, (10.4) becomes

$$0 = \sum_{\sigma \in \mathcal{L}_p} a_\sigma x_p + \sum_{\substack{p \lhd q = \tau(p): \\ \tau \notin \mathcal{L}_p}} a_\tau d_q \sqrt{\lambda} + \mathcal{O}\left(|x_p|^2 + |\lambda| + |x_p| \cdot \sqrt{|\lambda|} \right).$$

Since p is non-critical, $\sum_{\sigma \in \mathcal{L}_p} a_\sigma \neq 0$. Thus, by the implicit function theorem, we obtain that this equation is uniquely solved by

$$x_p(\lambda) = -\frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)}}{\sum_{\sigma \in \mathcal{L}_p} a_\sigma} \cdot \sqrt{\lambda} + \mathcal{O}(|\lambda|).$$

The second representation of the coefficient follows from (10.5) which completes the proof. $\square$

Lemma 10.7 (subcritical case)**.** *Let p be non-maximal. Assume for all $q \rhd p$*

$$x_q(\lambda) = d_q \cdot \sqrt{-\lambda} + \mathcal{O}(|\lambda|)$$

for small $\lambda < 0$ and some $d_q \in \mathbb{R} \setminus \{0\}$. Then

$$x_p(\lambda) = D_p \cdot \sqrt{-\lambda} + \mathcal{O}(|\lambda|)$$

for small $\lambda < 0$, where

$$D_p = -\frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)}}{\sum_{\sigma \in \mathcal{L}_p} a_\sigma} = \frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)}}{\sum_{\tau \notin \mathcal{L}_p} a_\tau} \neq 0.$$

Proof. The proof is completely analogous to the previous one. $\square$

The maximal cells simultaneously determine whether a bifurcation in the entire network occurs super- or subcritically. The non-maximal cells have no further influence, as we have seen in the previous two lemmas. We may, therefore, perform an inductive proof with respect to $\trianglelefteq$, summarizing Lemmas 10.4, 10.6 and 10.7, to obtain

Theorem 10.8. *Under the bifurcation assumption* (B) *and assuming the maximal cells in C to be critical, the state variables x_p bifurcate according to one of the following two cases.*

(i) *(supercritical)*

$$x_p(\lambda) = D_p \cdot \sqrt{\lambda} + \mathcal{O}(|\lambda|) \quad \textit{for small} \quad \lambda > 0, \quad \textit{if} \quad \frac{\ell}{\sum_{\sigma,\tau \in \Sigma} f_{\sigma\tau}} < 0;$$

(ii) *(subcritical)*

$$x_p(\lambda) = D_p \cdot \sqrt{-\lambda} + \mathcal{O}(|\lambda|) \quad \textit{for small} \quad \lambda < 0, \quad \textit{if} \quad \frac{\ell}{\sum_{\sigma,\tau \in \Sigma} f_{\sigma\tau}} > 0.$$

Therein the coefficients D_p generically do not vanish. They are defined recursively. For p maximal they are

(i) *(supercritical)*

$$D_p \in \left\{ +\sqrt{-\frac{\ell}{\sum_{\sigma,\tau \in \Sigma} f_{\sigma\tau}}}, -\sqrt{-\frac{\ell}{\sum_{\sigma,\tau \in \Sigma} f_{\sigma\tau}}} \right\};$$

(ii) *(subcritical)*

$$D_p \in \left\{ +\sqrt{\frac{\ell}{\sum_{\sigma,\tau \in \Sigma} f_{\sigma\tau}}}, -\sqrt{\frac{\ell}{\sum_{\sigma,\tau \in \Sigma} f_{\sigma\tau}}} \right\}.$$

The remaining ones are defined via

$$D_p = -\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)}}{\sum_{\sigma\in\mathcal{L}_p} a_\sigma} = \frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)}}{\sum_{\tau\notin\mathcal{L}_p} a_\tau}.$$

Globally, there are 2^m, with $m = \#\{p \in C \mid p \text{ maximal}\}$, different branches of steady states in both cases that are determined by the choices of D_p for p maximal.

The branching solutions in Theorem 10.8 are the same as the ones described in Proposition 5.1 in SOARES [99] for layered feedforward networks (compare to Remark 8.37) and in Proposition 5.7 of AGUIAR, DIAS, and SOARES [12] investigating feedforward structure of transitive components. Here we extend these results by the explicit computation of the lowest order coefficients.

Remark 10.9. Note that by restriction to the invariant fully synchronous subspace, we obtain a fully synchronous saddle node bifurcation similar to the proof of Lemma 10.4. This will be made more precise in the case of non-critical maximal cells (see Lemma 10.17 in Section 10.2) where it is of great importance. Hence, two of the branching solutions provided by Theorem 10.8 necessarily describe this branch. It can easily be seen from the recursive formulas using (10.5) that these are exactly the ones where the coefficients of all maximal cells have the same sign. This is also to be expected, as in all other cases not even the maximal cells are synchronous. △

Remark 10.10. The results of Theorem 10.8 are only accurate if none of the D_p vanish. This can be seen to be true generically. The system parameter ℓ does not vanish generically. Hence, $D_p \neq 0$ for p maximal. In particular, the coefficients of the fully synchronous branching solutions do not vanish generically. Hence, consider a branch for which the coefficients D_p for p maximal have different signs. From the inductive formulas in both cases, we see that the coefficient D_p for a non-maximal cell p is a fraction with denominator a finite product of terms $\sum_{\sigma\notin\mathcal{L}_q} a_\sigma$ times D_p for p maximal and nominator a polynomial in the a_σ. Hence, it suffices to show that the nominator does not identically vanish for all values of the a_σ. To that end recall from Corollary 8.24 that there is an input map $\kappa \in \Sigma$ such that $\kappa(p)$ is maximal for all $p \in C$. Let $a_{\mathrm{Id}} = -1, a_\kappa = 1$ and all the other $a_\sigma = 0$. Then $\sum_{\sigma\in\Sigma} a_\sigma = 0$ so that this is a suitable choice of coefficients for critical maximal cells. Note that $\mathrm{Id} \in \mathcal{L}_p$ for all $p \in C$ so that a_{Id} is not a part of any of the terms in the recursive formula (in its second formulation). For p non-maximal $\kappa \notin \mathcal{L}_p$ and we compute $D_p = D_{\kappa(p)} \neq 0$ as $\kappa(p)$ is maximal. That means the coefficients do not vanish identically for all choices of a_σ which implies that they do not vanish for a generic choice. This observation holds true independent of the direction of branching – i.e. it holds for super- and subcritical bifurcations. △

10.2 Maximal cells are non-critical

Next, we assume that the maximal cells are non-critical. In particular, $\sum_{\sigma\in\Sigma} a_\sigma \neq 0$ under the condition of genericity. The general strategy for finding branching solutions remains the same as in the previous part. We solve (10.4) for a given cell p assuming knowledge of its inputs. However, the considerations, especially for non-maximal cells, become more involved, as we have to distinguish whether a cell is critical in each step. Inductively, this provides all possible solutions for all cells. As a result of the multitude of different cases, the explicit bifurcation patterns – governed by simultaneous solutions of (10.4) for all cells – become a lot more complex than in the previous case. Once again, we have to distinguish branching solutions that exist for positive and negative values of the bifurcation parameter λ. We refer to these cases as *super-* and *subcritical*, as before. The result for the subcritical case, however, can be obtained as a corollary from the supercritical case, as we see in Theorem 10.21.

10.2.1 Solving Equation (10.4)

We begin by solving (10.4) in different settings: first for maximal cells, then for non-maximal cells under specific assumptions on the branching solutions for cells above with respect to $\trianglelefteq$. This allows to set up the inductive proof of the bifurcation result for the entire network.

Lemma 10.11. *Let $p \in C$ be maximal. Equation* (10.4) *generically has a unique solution*

$$x_p(\lambda) = D_p\lambda + \mathcal{O}(|\lambda|^2)$$

for $|\lambda|$ small with

$$D_p = -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma} \neq 0.$$

Proof. Equation (10.4) becomes

$$0 = \sum_{\sigma\in\Sigma} a_\sigma x_p + \ell\lambda + \mathcal{O}(|x_p|^2 + |x_p||\lambda| + |\lambda|^2).$$

The implicit function theorem yields a unique branch of steady states with

$$x_p(\lambda) = -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}\lambda + \mathcal{O}(|\lambda|^2),$$

for small absolute values of λ, since $\sum_{\sigma\in\Sigma} a_\sigma \neq 0$. Note that $\ell \neq 0$ generically as well which completes the proof. □

Remark 10.12. The non-critical maximal cells do not undergo a bifurcation. The steady state solution to their respective ordinary differential equation persists for all parameter values close to zero depending smoothly on λ. This holds true simultaneously for all maximal cells. △

Similar to Section 10.1, we investigate the non-maximal cells inductively. We focus on $\lambda \geq 0$ and assume the following input scenarios for a fixed non-maximal cell $p \in C$ as a hypothesis:

(H) For all $q \rhd p$ and small $\lambda > 0$ the state variable evolves as

$$x_q(\lambda) = d_q \cdot \lambda^{2^{-\xi_q}} + \mathcal{O}\left(|\lambda|^{2^{-(\xi_q-1)}}\right),$$

where $d_q \in \mathbb{R}\setminus\{0\}$ and the ξ_q are integers with $0 \leq \xi_q$ that define the square root order of the branching solution of cell q.

Under the assumption **(H)** we define

$$\xi_p = \max_{q\rhd p} \xi_q, \tag{10.7}$$

to be the largest square root order of inputs into cell p. Then, $\xi_p = 0$ if and only if all inputs into cell p evolve linearly in λ up to lowest order. To further simplify we denote the set of cells $q \rhd p$ which are of highest possible square root order in λ by Q_p. That is

$$Q_p = \left\{ q \rhd p \;\middle|\; x_q(\lambda) = d_q \cdot \lambda^{2^{-\xi_p}} + \mathcal{O}\left(|\lambda|^{2^{-(\xi_p-1)}}\right) \right\}. \tag{10.8}$$

In the case $\xi_p = 0$ all state variables x_q for $q \rhd p$ evolve linearly in λ up to lowest order. Hence, $Q_p = \{q \rhd p\}$. Note that ξ_p and Q_p are only defined for p non-maximal.

The case $\xi_p = 0$ has to be treated slightly differently than the others. More precisely, we need to distinguish two cases

(L)

$$\sum_{\tau\notin\mathcal{L}_p} a_\tau d_{\tau(p)} + \ell \neq 0,$$

(T)

$$\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} + \ell = 0.$$

In all other cases – i.e. for $\xi_p > 0$ – we formulate the following non-degeneracy condition:

(SN)

$$\sum_{\tau:\, \tau(p) \in Q_p} a_\tau d_{\tau(p)} \neq 0.$$

The conditions (L) and (T) in the case $\xi_p = 0$ and (SN) in all other cases describe whether the lowest order terms in λ in (10.4) vanish. We will see later (Theorem 10.20) that we may focus on (L) for non-critical cells with $\xi_p = 0$. On the other hand, the seemingly non-generic condition (T) turns out to be generically satisfied, for critical cells with $\xi_p = 0$. Furthermore, (SN) can generically be assumed to be true, when $\xi_p > 0$.

Under the given assumptions, we prove statements providing branching solutions to (10.4) for non-maximal cells. We start with the case that p non-maximal is non-critical.

Lemma 10.13. *Let $p \in C$ be non-maximal and non-critical. Under assumption* (**H**) *with additional assumption* (L) *or* (SN)*, if $\xi_p = 0$ or $\xi_p > 0$ respectively,* (10.4) *has the unique solution*

$$x_p(\lambda) = d_p \cdot \lambda^{2^{-\xi_p}} + \mathcal{O}\left(|\lambda|^{2^{-(\xi_p - 1)}}\right)$$

for $\lambda > 0$ small, where $d_p \neq 0$ and

$$d_p = \begin{cases} -\dfrac{\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} + \ell}{\sum_{\sigma \in \mathcal{L}_p} a_\sigma}, & \textit{if} \quad \xi_p = 0; \\ -\dfrac{\sum_{\tau:\, \tau(p) \in Q_p} a_\tau d_{\tau(p)}}{\sum_{\sigma \in \mathcal{L}_p} a_\sigma}, & \textit{if} \quad \xi_p > 0. \end{cases}$$

Proof. The proofs for both cases are very similar and analogous to the proofs for Lemmas 10.6 and 10.7. Hence, we only sketch them here. We assume (**H**), $\xi_p = 0$, and (L) first. Equation (10.4) becomes

$$0 = \sum_{\sigma \in \mathcal{L}_p} a_\sigma x_p + \sum_{\substack{p \lhd q = \tau(p): \\ \tau \notin \mathcal{L}_p}} a_\tau d_q \lambda + \ell\lambda + \mathcal{O}\left(|x_p|^2 + |\lambda||x_p| + |\lambda|^2\right).$$

As $\sum_{\sigma \in \mathcal{L}_p} a_\sigma \neq 0$ this is uniquely solved by

$$x_p(\lambda) = -\frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} + \ell}{\sum_{\sigma \in \mathcal{L}_p} a_\sigma} \cdot \lambda + \mathcal{O}\left(|\lambda|^2\right)$$

for small $\lambda > 0$, due to the implicit function theorem. Note that, because of assumption (L), the linear coefficient does not vanish.

Next, assume (**H**), $\xi_p > 0$, and (SN). Equation (10.4) becomes

$$0 = \sum_{\sigma \in \mathcal{L}_p} a_\sigma x_p + \sum_{\substack{\tau \notin \mathcal{L}_p: \\ \tau(p) = q \in Q_p}} a_\tau d_q \lambda^{2^{-\xi_p}} + \mathcal{O}\left(|x_p|^2 + |\lambda|^{2^{-\xi_p}}|x_p| + |\lambda|^{2^{-(\xi_p - 1)}}\right).$$

By the same argument as before, this is uniquely solved by

$$x_p(\lambda) = -\frac{\sum_{\tau:\, \tau(p) \in Q_p} a_\tau d_{\tau(p)}}{\sum_{\sigma \in \mathcal{L}_p} a_\sigma} \cdot \lambda^{2^{-\xi_p}} + \mathcal{O}\left(|\lambda|^{2^{-(\xi_p - 1)}}\right)$$

for small $\lambda > 0$ with non-vanishing lowest order coefficient. □

The corresponding result for critical non-maximal cells is proved in multiple lemmas. We distinguish between the cases (L), (T) and (SN) for corresponding values of ξ_p.

Lemma 10.14. *Let $p \in C$ be non-maximal and critical. Assume* (**H**) *to hold true. Furthermore, assume $\xi_p = 0$ and the additional assumption* (L) *to hold true as well. The solutions to* (10.4) *generically bifurcate as in one of the following two cases:*

(i) If

$$\frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} + \ell}{\sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau}} > 0,$$

there are no branching solutions.

(ii) If

$$\frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} + \ell}{\sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau}} < 0,$$

the solutions undergo a saddle node bifurcation

$$x_p(\lambda) = d_p^{\pm} \cdot \sqrt{\lambda} + \mathcal{O}\left(|\lambda|\right)$$

for $\lambda > 0$ small, where

$$d_p^{\pm} = \pm\sqrt{-\frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} + \ell}{\sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau}}} \neq 0.$$

Proof. The proof is analogous to the one for Lemma 10.4, except for slightly varied coefficients. Once again it uses the standard technique for detecting saddle node bifurcations as in Murdock [75]. As $\xi_p = 0$, (10.4) becomes

$$0 = \sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau} x_p^2 + \sum_{\substack{\tau \notin \mathcal{L}_p \\ p \lhd q = \tau(p)}} a_\tau d_q \lambda + \ell\lambda + \mathcal{O}\left(|x_p|^3 + |\lambda||x_p| + |\lambda|^2\right).$$

We introduce a new variable $x_p = \mu y$ where $\mu = \sqrt{\lambda}$ for small $\lambda > 0$. The equation to be solved transforms into

$$0 = \sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau} \mu^2 y^2 + \sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} \mu^2 + \ell\mu^2 + \mathcal{O}\left(|\mu|^3|y| + |\mu|^4\right).$$

As $\mu > 0$, we may divide by μ^2 and obtain

$$0 = \sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau} y^2 + \sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} + \ell + \mathcal{O}\left(|\mu||y| + |\mu|^2\right) = g(y,\mu).$$

If $\left(\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} + \ell\right) / \sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau} > 0$, the equation $g(y,0) = 0$ has no real solutions. If, on the other hand, $\left(\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} + \ell\right) / \sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau} < 0$, there are two solutions to $g(y,0) = 0$

$$\overline{y}^{\pm} = \pm\sqrt{-\frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} + \ell}{\sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau}}}.$$

Furthermore, $\frac{\partial}{\partial y} g(\overline{y}^{\pm}, 0) = 2\sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau} \overline{y}^{\pm}$, which does not vanish, due to assumption (L). Hence, by the implicit function theorem, we obtain two branches of solutions

$$\overline{Y}^{\pm}(\mu) = \overline{y}^{\pm} + \mathcal{O}(|\mu|).$$

Transforming back into the original variables, we obtain the two branches

$$x_p(\lambda) = \pm\sqrt{-\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau d_{\tau(p)} + \ell}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}\sqrt{\lambda} + \mathcal{O}(|\lambda|)}$$

for small $\lambda > 0$. $\square$

In order to formulate the corresponding result in the case (T), we refine the hypothesis (**H**) only in the case $\xi_p = 0$ to be

(**H**) For all $q \rhd p$ and small $\lambda > 0$ the state variable evolves as

$$x_q(\lambda) = d_q \cdot \lambda + R_q \cdot \lambda^2 + \mathcal{O}\left(|\lambda|^3\right),$$

where $d_q, R_q \in \mathbb{R}$ and $d_q \neq 0$.

The only difference is the specification of the second order coefficient R_q, which is not needed for the computations whenever $\xi_p > 0$.

Lemma 10.15. *Let $p \in C$ be non-maximal and critical. Assume the induction assumption* (**H**) *to hold true. Furthermore, assume $\xi_p = 0$ and the additional assumption* (T) *to hold true as well. Define*

$$\begin{aligned}
A &= \sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau},\\
B &= \sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda} + 2\sum_{\sigma\in\mathcal{L}_p,\tau\notin\mathcal{L}_p} f_{\sigma\tau} d_{\tau(p)},\\
C &= \sum_{\tau\notin\mathcal{L}_p} a_\tau R_{\tau(p)} + \sum_{\tau\notin\mathcal{L}_p} f_{\tau\lambda} d_{\tau(p)} + \sum_{\sigma,\tau\notin\mathcal{L}_p} f_{\sigma\tau} d_{\sigma(p)} d_{\tau(p)} + f_{\lambda\lambda}.
\end{aligned}$$

The solutions to (10.4) *generically bifurcate as in one of the following two cases:*

(i) If

$$B^2 - 4AC < 0,$$

there are no branching solutions.

(ii) If

$$B^2 - 4AC > 0,$$

the solutions undergo a transcritical bifurcation

$$x_p(\lambda) = d_p^\pm \lambda + \mathcal{O}\left(|\lambda|^2\right),$$

for $\lambda > 0$ small, where

$$d_p^\pm = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \neq 0.$$

Proof. Under the assumptions (10.4) becomes

$$\begin{aligned}
0 &= \sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau} x_p^2 + \sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda}\lambda x_p + \underbrace{\sum_{\substack{\tau\notin\mathcal{L}_p\\ p\lhd q=\tau(p)}} a_\tau d_q \lambda + \ell\lambda}_{=0} + \sum_{\substack{\tau\notin\mathcal{L}_p\\ p\lhd q=\tau(p)}} a_\tau R_q \lambda^2\\
&\quad + \sum_{\substack{\tau\notin\mathcal{L}_p\\ p\lhd q=\tau(p)}} f_{\tau\lambda} d_q \lambda^2 + 2\sum_{\substack{\sigma\in\mathcal{L}_p\\ \tau\notin\mathcal{L}_p:\\ p\lhd q=\tau(p)}} f_{\sigma\tau} d_q \lambda x_p + \sum_{\substack{\sigma,\tau\notin\mathcal{L}_p\\ p\lhd q, s=\sigma(p),\tau(p)}} f_{\sigma\tau} d_q d_s \lambda^2 + f_{\lambda\lambda}\lambda^2\\
&\quad + \mathcal{O}\left(|x_p|^3 + |x_p|^2|\lambda| + |x_p||\lambda|^2 + |\lambda|^3\right)\\
&= A x_p^2 + B\lambda x_p + C\lambda^2 + \mathcal{O}\left(|x_p|^3 + |x_p|^2|\lambda| + |x_p||\lambda|^2 + |\lambda|^3\right).
\end{aligned}$$

Similar to the previous proofs, we employ the standard method to detect transcritical bifurcations (see for example MURDOCK [75]) by introducing a new variable $x_p = \lambda y$. The equation becomes

$$\begin{aligned} 0 &= A\lambda^2 y^2 + B\lambda^2 y + C\lambda^2 + \mathcal{O}\left(|y||\lambda|^3 + |\lambda|^3\right) \\ &= \lambda^2 \left(Ay^2 + By + C + \mathcal{O}\left(|y||\lambda| + |\lambda|\right)\right) = \lambda^2 g(y, \lambda). \end{aligned}$$

Note that the coefficients A, B and C are composed of the second order partial derivatives of f. Hence, generically $A, B^2 - 4AC \neq 0$. If $B^2 - 4AC < 0$ there are no solutions to $g(y, 0) = 0$ – this proves the first case. If, on the other hand, $B^2 - 4AC > 0$ we obtain two solutions:

$$\overline{y}^{\pm} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Furthermore, $\frac{\partial}{\partial y} g(\overline{y}^{\pm}, 0) = 2A\overline{y}^{\pm} + B$ which, generically, does not vanish. Hence, by the implicit function theorem, we obtain two branches of solutions

$$\overline{Y}^{\pm}(\lambda) = \overline{y}^{\pm} + \mathcal{O}(|\lambda|)$$

for small $\lambda > 0$. Transforming back into the original coordinates, we obtain

$$x_p(\lambda) = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \cdot \lambda + \mathcal{O}\left(|\lambda|^2\right),$$

which completes the proof. □

Finally, we turn to the branching solutions for critical cells p with $\xi_p > 0$ with the additional non-degenericity condition (SN).

Lemma 10.16. *Let $p \in C$ be non-maximal and critical. Assume the induction hypothesis* (**H**) *to hold true. Furthermore, assume $\xi_p > 0$ and the additional non-degenericity condition* (SN) *to hold true as well. The solutions to* (10.4) *generically bifurcate as in one of the following two cases:*

(i) If

$$\frac{\sum_{\tau:\, \tau(p) \in Q_p} a_\tau d_{\tau(p)}}{\sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau}} > 0,$$

there are no branching solutions.

(ii) If

$$\frac{\sum_{\tau:\, \tau(p) \in Q_p} a_\tau d_{\tau(p)}}{\sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau}} < 0,$$

the solutions undergo a saddle node bifurcation as

$$x_p(\lambda) = d_p^{\pm} \cdot \lambda^{2^{-(\xi_p+1)}} + \mathcal{O}\left(|\lambda|^{2^{-\xi_p}}\right),$$

for $\lambda > 0$ small, where

$$d_p^{\pm} = \pm\sqrt{-\frac{\sum_{\tau:\, \tau(p) \in Q_p} a_\tau d_{\tau(p)}}{\sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau}}} \neq 0.$$

Proof. Under the given assumptions (10.4) becomes

$$0 = \sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau} x_p^2 + \sum_{\substack{\tau \notin \mathcal{L}_p \\ \tau(p) = q \in Q_p}} a_\tau d_q \lambda^{2^{-\xi_p}} + \mathcal{O}\left(|x_p|^3 + |x_p||\lambda|^{2^{-\xi_p}} + |\lambda|^{2^{-(\xi_p-1)}}\right).$$

Similar to previous proofs, we introduce new coordinates $x_p = \mu y$, where $\mu = \sqrt{\lambda^{2^{-\xi_p}}} = \lambda^{2^{-(\xi_p+1)}}$ for small $\lambda > 0$. The equation becomes

$$0 = \sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau}\mu^2 y^2 + \sum_{\substack{\tau \notin \mathcal{L}_p \\ \tau(p)=q \in Q_p}} a_\tau d_q \mu^2 + \mathcal{O}\left(|y||\mu|^3 + |\mu|^4\right).$$

As $\mu > 0$, we may divide by μ^2 to obtain

$$0 = \sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau} y^2 + \sum_{\substack{\tau \notin \mathcal{L}_p \\ \tau(p)=q \in Q_p}} a_\tau d_q + \mathcal{O}\left(|y||\mu| + |\mu|^2\right) = g(y, \mu).$$

If $\left(\sum_{\tau\colon \tau(p) \in Q_p} a_\tau d_{\tau(p)}\right) / \sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau} > 0$ there are no solutions to $g(y, 0) = 0$ – this proves the first case. If, on the other hand, $\left(\sum_{\tau\colon \tau(p) \in Q_p} a_\tau d_{\tau(p)}\right) / \sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau} < 0$, there are two solutions to $g(y, 0) = 0$, as $\sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau} \neq 0$ generically:

$$\overline{y}^{\pm} = \pm\sqrt{-\frac{\sum_{\tau\colon \tau(p) \in Q_p} a_\tau d_{\tau(p)}}{\sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau}}}.$$

Furthermore, $\frac{\partial}{\partial y} g(\overline{y}^{\pm}, 0) = 2\sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau}\overline{y}^{\pm}$, which, by the same argument, generically does not vanish. Hence, by the implicit function theorem, we obtain two branches of solutions

$$\overline{Y}^{\pm}(\mu) = \overline{y}^{\pm} + \mathcal{O}(|\mu|).$$

Transforming back into the original coordinates, we obtain

$$x_p(\lambda) = \pm\sqrt{-\frac{\sum_{\tau\colon \tau(p) \in Q_p} a_\tau d_{\tau(p)}}{\sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau}}} \cdot \lambda^{2^{-(\xi_p+1)}} + \mathcal{O}\left(|\lambda|^{2^{-\xi_p}}\right),$$

for small $\lambda > 0$, which completes the proof. □

10.2.2 Branches of steady states for the entire network

We have now collected all the pieces that are needed for the inductive proof of existence – and non-existence – of branching solutions to the bifurcation problem (B) with non-critical maximal cells. This, in turn, allows us to describe the complete branching pattern. Some subtleties arise while investigating which cases from Lemmas 10.11 and 10.13 to 10.16 can generically occur, when (10.4) is solved for all $p \in C$ simultaneously, and in proving that, generically, there are no other possible solutions. We investigate a specific fully synchronous branch of steady states first.

Lemma 10.17. *Under the bifurcation assumption* (B) *with non-critical maximal cells, there exists a fully synchronous branch of steady states*

$$x_p(\lambda) = X(\lambda) = D\lambda + R\lambda^2 + \mathcal{O}\left(|\lambda|^3\right) \tag{10.9}$$

for $|\lambda|$ small and all $p \in C$. Therein

$$D = -\frac{\ell}{\sum_{\sigma \in \Sigma} a_\sigma}, \tag{10.10}$$

$$R = -\frac{\sum_{\sigma,\tau \in \Sigma} f_{\sigma\tau}\ell^2 - \sum_{\sigma \in \Sigma} a_\sigma \sum_{\sigma \in \Sigma} f_{\sigma\lambda}\ell + \left(\sum_{\sigma \in \Sigma} a_\sigma\right)^2 f_{\lambda\lambda}}{\left(\sum_{\sigma \in \Sigma} a_\sigma\right)^3}, \tag{10.11}$$

which generically do not vanish.

Proof. Consider the fully synchronous subspace $\Delta_0 = \{x_{p_1} = \ldots = x_{p_N}\} \subset \bigoplus_{p \in C} \mathbb{R}$. This can readily be seen to be invariant under the flow induced by network vector fields of the form (10.1). Choosing a coordinate y for this subspace, the bifurcation problem becomes the same for all cells $p \in C$. The Taylor expanded equation (10.4) is

$$0 = \sum_{\sigma \in \Sigma} a_\sigma y + \ell \lambda + \mathcal{O}(|y|^2 + |y||\lambda| + |\lambda|^2).$$

This is precisely the same equation as the one in the proof of Lemma 10.11. With the same argumentation we obtain a unique branch of solutions

$$x_p(\lambda) = y(\lambda) = -\frac{\ell}{\sum_{\sigma \in \Sigma} a_\sigma} \lambda + \mathcal{O}\left(|\lambda|^2\right)$$

for $f(y, \ldots, y, \lambda) = 0$ with $|\lambda|$ small. Performing second order implicit derivation – for which we omit the details –, we compute

$$y''(\lambda) = -2 \cdot \frac{\sum_{\sigma,\tau \in \Sigma} f_{\sigma\tau} \ell^2 - \sum_{\sigma \in \Sigma} a_\sigma \sum_{\sigma \in \Sigma} f_{\sigma\lambda} \ell + \left(\sum_{\sigma \in \Sigma} a_\sigma\right)^2 f_{\lambda\lambda}}{\left(\sum_{\sigma \in \Sigma} a_\sigma\right)^3}.$$

As both derivatives generically do not vanish, this completes the proof. □

Furthermore, we need the following technical statements whose proofs consist of technical computations only. Hence, we prove them as Lemmas B.1 and B.2 in Appendix B.

Lemma 10.18. *Let $p \in C$ be non-maximal and non-critical. Assume*

$$d_q = -\frac{\ell}{\sum_{\sigma \in \Sigma} a_\sigma},$$

for all $q \rhd p$. Then

$$-\frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} + \ell}{\sum_{\sigma \in \mathcal{L}_p} a_\sigma} = -\frac{\ell}{\sum_{\sigma \in \Sigma} a_\sigma}.$$

In particular, this computation holds for the coefficient d_p in Lemma 10.13 if all $q \rhd p$ bifurcate according to the fully synchronous state.

Lemma 10.19. *Let $p \in C$ be non-maximal and critical. Assume* (**H**) *with*

$$d_q = -\frac{\ell}{\sum_{\sigma \in \Sigma} a_\sigma}, \qquad R_q = -\frac{\sum_{\sigma,\tau \in \Sigma} f_{\sigma\tau} \ell^2 - \sum_{\sigma \in \Sigma} a_\sigma \sum_{\sigma \in \Sigma} f_{\sigma\lambda} \ell + \left(\sum_{\sigma \in \Sigma} a_\sigma\right)^2 f_{\lambda\lambda}}{\left(\sum_{\sigma \in \Sigma} a_\sigma\right)^3}$$

for all $q \rhd p$ and define

$$\begin{aligned}
A &= \sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau}, \\
B &= \sum_{\sigma \in \mathcal{L}_p} f_{\sigma\lambda} + 2 \sum_{\sigma \in \mathcal{L}_p, \tau \notin \mathcal{L}_p} f_{\sigma\tau} d_{\tau(p)}, \\
C &= \sum_{\tau \notin \mathcal{L}_p} a_\tau R_{\tau(p)} + \sum_{\tau \notin \mathcal{L}_p} f_{\tau\lambda} d_{\tau(p)} + \sum_{\sigma,\tau \notin \mathcal{L}_p} f_{\sigma\tau} d_{\sigma(p)} d_{\tau(p)} + f_{\lambda\lambda}, \\
E &= \sum_{\sigma \in \mathcal{L}_p} f_{\sigma\lambda} - 2 \cdot \frac{\ell}{\sum_{\sigma \in \Sigma} a_\sigma} \sum_{\substack{\sigma \in \Sigma \\ \tau \in \mathcal{L}_p}} f_{\sigma\tau}.
\end{aligned}$$

Then generically

$$B^2 - 4AC = E^2 > 0$$

and

$$\frac{-B+E}{2A} = -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}, \qquad \frac{-B-E}{2A} = M$$

with

$$M = \frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma} \cdot \left(1 + 2\frac{\sum_{\sigma\in\mathcal{L}_p, \tau\notin\mathcal{L}_p} f_{\sigma\tau}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}\right) - \frac{\sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}.$$

In particular, these computations hold for the coefficients $d_p^\pm$ in Lemma 10.15 if all $q \rhd p$ bifurcate according to the fully synchronous state.

The main result follows inductively from Lemmas 10.11 and 10.13 to 10.16, by carefully investigating which cases and assumptions can hold true, if all equations are solved simultaneously. To that end recall that $\Omega_{\overline{p},p}$ denotes the set of all paths from $\overline{p}$ to p without any loops (Definition 3.2) and that a block is a subset of cells $B \subset C$ that does not receive any inputs from outside of B, i.e. $\sigma(B) \subset B$ for all $\sigma \in \Sigma$ (Definition 3.34).

Theorem 10.20 (Supercritical branches)**.** *Under the bifurcation assumption* (B) *with non-critical maximal cells, generically,*[1] *all bifurcating branches of steady states for $\lambda > 0$ are as follows: There is a maximal block $B \subset C$ such that the state variable x_p for each $p \in B$ remains in the fully synchronous state $x_p(\lambda) = X(\lambda)$ as in* (10.9)*. Maximality means that there is no block with the same properties containing B.*

The state variables of all $p \notin B$ bifurcate as

$$x_p(\lambda) = D_p \lambda^{2^{-\mu_p}} + \mathcal{O}\left(|\lambda|^{2^{-(\mu_p-1)}}\right),$$

where $D_p \neq 0$ and

$$\mu_p = \max_{\overline{p}\in B} \max_{\omega\in\Omega_{\overline{p},p}} \#\{q \in \omega \mid q \textit{ critical}, q \notin B\} - 1$$

is the maximal number of critical cells $q \notin B$ along paths from any cell in B to p.

The branch exists, if and only if

$$\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} + \ell}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}} < 0, \qquad \frac{\sum_{\tau:\,\tau(p)\in Q_p} a_\tau D_{\tau(p)}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}} < 0$$

for all critical cells p with $\mu_p = 1$ or $\mu_p > 1$ respectively. Therein, Q_p is defined analogous to (10.8) *with ξ_p replaced by μ_p.*

The D_p are chosen according to the inductive rules:

D_p	*Case*
$-\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} + \ell}{\sum_{\sigma\in\mathcal{L}_p} a_\sigma}$	p *non-critical*, $\mu_p = 0$
$-\frac{\sum_{\tau:\,\tau(p)\in Q_p} a_\tau D_{\tau(p)}}{\sum_{\sigma\in\mathcal{L}_p} a_\sigma}$	p *non-critical*, $\mu_p > 0$
$\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma} \cdot \left(1 + 2\frac{\sum_{\sigma\in\mathcal{L}_p,\tau\notin\mathcal{L}_p} f_{\sigma\tau}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}\right) - \frac{\sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}$	p *critical*, $\mu_p = 0$
$\pm\sqrt{-\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} + \ell}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}}$	p *critical*, $\mu_p = 1$
$\pm\sqrt{-\frac{\sum_{\tau:\,\tau(p)\in Q_p} a_\tau D_{\tau(p)}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}}$	p *critical*, $\mu_p > 1$

[1] i.e. for an open and dense subset of system parameters $\{a_\sigma, f_{\sigma\tau}, f_{\sigma\lambda}, \ell, f_{\lambda\lambda} \mid \sigma,\tau \in \Sigma\}$

Proof. First, let us note that the fully synchronous branch of steady states (see Lemma 10.17) is covered by the statement of the theorem for $\lambda > 0$. In that case $B = C$, which is obviously a block, and there are no further cells for which coefficients D_p have to be chosen. Hence, for the remainder of the proof we may assume $B \subsetneq C$. It consists of two inductive proofs that have overlaps in some of the computations. The first constructs a block B containing cells that remain in the fully synchronous state. The second iteratively investigates the remaining cells. It shows that the corresponding state variables bifurcate as stated in the proposition and that, generically, no other branching solutions exist. Throughout the proof, we indicate major steps for cross-referencing. Furthermore, we refer to cells being 'above' or 'below' others. This is meant to be understood in the sense of the partial order $\trianglelefteq$.

Construction of B For computational reasons we construct a second maximal block $\mathbb{B}$ containing those cells p that branch as

$$x_p(\lambda) = -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}\lambda + \mathcal{O}\left(|\lambda|^2\right).$$

This implies $B \subset \mathbb{B}$ but not necessarily that all cells in $\mathbb{B}$ branch synchronously, due to the higher order terms. Eventually, however, we see that the other inclusion holds as well.

Step 1. As was shown in Lemma 10.11, the state variables x_p for p maximal have unique solutions. By Lemma 10.17, this must be the same as in the fully synchronous state – the computation in Lemma 10.11 for D_p also matches. Hence, the maximal cells are always contained in B and $\mathbb{B}$.

Step 2. We consider a non-maximal cell $p \in C$ such that $q \in B \subset \mathbb{B}$ for all $q \triangleright p$. Such a cell exists due to the finiteness of C and the fact that $\trianglelefteq$ is a partial order. There are two possible cases: p can be critical or non-critical. Let us consider the latter case first. If p is non-critical

$$\sum_{\tau\notin\mathcal{L}_p} a_\tau \neq \sum_{\sigma\in\Sigma} a_\sigma$$

generically. Thus, using $\tau(p) \triangleright p$ for all $\tau \notin \mathcal{L}_p$, we obtain

$$\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} = -\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau}{\sum_{\sigma\in\Sigma} a_\sigma}\ell \neq -\ell$$

and condition (L) is fulfilled. By Lemma 10.13, we know that (10.4) has a unique solution for such p

$$x_p(\lambda) = D_p\lambda + \mathcal{O}\left(|\lambda|^2\right).$$

In Lemma 10.18 we show

$$D_p = -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}.$$

As we assume $\mathbb{B}$ to be maximal, we immediately obtain $p \in \mathbb{B}$. If, furthermore, p is such that all cells above p are maximal it receives the same input for all branches. Due to *Step* 1 the equation for maximal cells is uniquely solved by the fully synchronous state. Hence, there is no choice of coefficients D_q for $q \triangleright p$ leading to different inputs. As (10.4) has a unique solution for p, we know that this must be the fully synchronous one as well, i.e. $x_p(\lambda) = X(\lambda)$. Therefore, $p \in B$ in this case. Inductively, this consideration can be extended to any cell p such that all $q \triangleright p$ are non-critical by repeated application of this first part of *Step* 2.

Assume now, that p is critical, i.e. $\sum_{\sigma\in\mathcal{L}_p} a_\sigma = 0$. Then

$$\sum_{\tau\notin\mathcal{L}_p} a_\tau = \sum_{\sigma\in\Sigma} a_\sigma$$

which, using $\tau(p) \rhd p$ for $\tau \notin \mathcal{L}_p$, yields

$$\sum_{\tau \notin \mathcal{L}_p} a_\tau D_{\tau(p)} = -\frac{\sum_{\sigma\in\Sigma} a_\sigma}{\sum_{\sigma\in\Sigma} a_\sigma}\ell = -\ell$$

so that condition (T) is satisfied. As all $q \rhd p$ asymptotically grow as λ, all solutions of (10.4) for p are described by Lemma 10.15. Furthermore, using the computational results in Lemma 10.19, we are in the second case described in Lemma 10.15. This tells us that (10.4) generically has two solutions

$$x_p(\lambda) = D_p^{\pm}\lambda + \mathcal{O}\left(|\lambda|^2\right). \tag{10.12}$$

In Lemma 10.19 we show

$$\begin{aligned} D_p^+ &= -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}, \\ D_p^- &= \frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}\cdot\left(1 + 2\frac{\sum_{\sigma\in\mathcal{L}_p,\tau\notin\mathcal{L}_p} f_{\sigma\tau}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}\right) - \frac{\sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}. \end{aligned} \tag{10.13}$$

Generically, these two coefficients are not equal. Hence, choosing the positive sign in (10.12) yields $p \in \mathbb{B}$. The negative sign gives $p \notin \mathbb{B}$. Once again, if p is such that there are no critical cells above it, (10.4) is uniquely solved by the fully synchronous branch for all $q \rhd p$. Hence, there is no choice of coefficients D_q for which p receives different inputs – (10.4) for p is independent of the solution branch for the entire network. Therefore, the coefficient D_p^+ must provide the fully synchronous branch $x_p(\lambda) = X(\lambda)$ and we obtain $p \in B$ in this case.

Step 3. Next, we consider an arbitrary cell p that has critical cells above it. We assume that $q \rhd p$ critical implies $s \in B$ for all $s \rhd q$ and that all $q \rhd p$ branch as

$$x_q(\lambda) = D_q\lambda + \mathcal{O}\left(|\lambda|^2\right), \tag{10.14}$$

with

$$\begin{cases} D_q = -\dfrac{\sum_{\tau\notin\mathcal{L}_q} a_\tau D_{\tau(q)} + \ell}{\sum_{\sigma\in\mathcal{L}_q} a_\sigma} & \text{for } q \text{ non-critical}, \\ D_q \in \left\{-\dfrac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}, D_q^-\right\} & \text{for } q \text{ critical}, \end{cases}$$

where D_q^- is as in (10.13). Note that this assumption is valid for $q \in B$ maximal as well – then $\sum_{\tau\notin\mathcal{L}_q} a_\sigma = 0$. Furthermore, assume $q \in B$, if and only if either q is maximal or $s \in B$ for all $s \rhd q$, where, additionally, $D_q = -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}$, in the case that q is critical. Under these assumptions, we investigate the branching solutions of x_p and whether $p \in B$.

Given the assumptions above, we may partition the cells $q \rhd p$ according to three cases:

(i) $q \in B$,

(ii) $q \notin B$ critical,

(iii) $q \notin B$ non-critical.

We split $\Sigma \backslash \mathcal{L}_p = \mathcal{L}_0 \sqcup \mathcal{L}_1 \sqcup \mathcal{L}_2$ according to which of these cases $\tau(p)$ falls into for $\tau \notin \mathcal{L}_p$. In particular, for $\tau \in \mathcal{L}_0 \cup \mathcal{L}_1$, we have $s \in B$ for all $s \rhd \tau(p)$ according to the assumptions. Furthermore, note that for $\tau \in \mathcal{L}_2$ there exist $\eta \notin \mathcal{L}_{\tau(p)}$ such that $\eta(\tau(p)) \notin B$. Repeating this argument inductively for $\eta(\tau(p))$ we eventually reach a critical cell $\eta_1(\dots\eta_k(\tau(p))) \notin B$. Thus, setting $\eta = \eta_1 \cdots \eta_k$, we may assume $\eta(\tau(p)) \notin B$ to be critical – i.e. $\tau \in \mathcal{L}_2$ implies there exists $s \rhd \tau(p)$ such that $s \notin B$ and s is critical. In particular, this means $\eta\tau \in \mathcal{L}_1$ so that $\mathcal{L}_1 = \emptyset$ implies $\mathcal{L}_2 = \emptyset$.

The recursive assumption on the coefficients allows us to express D_q for all $q \rhd p$ in terms of the coefficients of cells in B – which all equal $-\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}$ – and possibly also of critical cells not in B if $q = \tau(p)$ and $\tau \in \mathcal{L}_2$. In particular, in the latter case we compute

$$D_q = -\frac{\sum_{\tau\notin\mathcal{L}_q} a_\tau D_{\tau(q)} + \ell}{\sum_{\sigma\in\mathcal{L}_q} a_\sigma} = -A_q \cdot \frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma} + \sum_{\substack{s\rhd q \\ s \text{ critical} \\ s\notin B}} B_q^s \cdot D_s^- + C_q \cdot \ell.$$

Therein the coefficients A_q, B_q^s, C_q are rational expressions in $a_{\sigma_1}, \dots, a_{\sigma_n}$ where the denominator is a product of terms of the form $\sum_{\sigma\in\mathcal{L}_q} a_\sigma$ that do not vanish. For $q = \tau(p)$ with $\tau \in \mathcal{L}_0 \cup \mathcal{L}_1$ we have that $s \in B$ for all $s \rhd q$ so that we may express D_q in terms of the coefficients of cells in B exclusively.

For the cell p in consideration we then obtain

$$\begin{aligned}
\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} &= -\sum_{\tau\in\mathcal{L}_0} a_\tau \frac{\ell}{\sum_{\sigma\in\Sigma}} + \sum_{\tau\in\mathcal{L}_1} a_\tau D^-_{\tau(p)} \\
&\quad + \sum_{\tau\in\mathcal{L}_2} a_\tau \left(-A_{\tau(p)} \frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma} + \sum_{\substack{s\rhd\tau(p) \\ s \text{ critical} \\ s\notin B}} B^s_{\tau(p)} D^-_s + C_{\tau(p)}\ell \right) \\
&= -\left(\sum_{\tau\in\mathcal{L}_0} a_\tau + \sum_{\tau\in\mathcal{L}_2} a_\tau A_{\tau(p)} \right) \frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma} + \sum_{\tau\in\mathcal{L}_2} a_\tau C_{\tau(p)} \ell \\
&\quad + \sum_{\tau\in\mathcal{L}_1} \left(a_\tau + \sum_{\substack{\eta\in\mathcal{L}_2 \\ \tau(p)\rhd\eta(p)}} a_\eta B^{\tau(p)}_{\eta(p)} \right) D^-_{\tau(p)},
\end{aligned} \tag{10.15}$$

where we have used that for $\eta \in \mathcal{L}_2$ and $s \rhd \eta(p)$ critical also $s \rhd p$ and there is $\tau \in \mathcal{L}_1$ such that $\tau(p) = s$ by transitivity of $\unlhd$. The sum in (10.15) generically equals $-\ell$ – i.e. condition (T) is fulfilled –, if and only if

$$-\frac{\left(\sum_{\tau\in\mathcal{L}_0} a_\tau + \sum_{\tau\in\mathcal{L}_2} a_\tau A_{\tau(p)}\right)}{\sum_{\sigma\in\Sigma} a_\sigma} + \sum_{\tau\in\mathcal{L}_2} a_\tau C_{\tau(p)} = -1 \tag{10.16}$$

$$\sum_{\tau\in\mathcal{L}_1} \left(a_\tau + \sum_{\substack{\eta\in\mathcal{L}_2 \\ \tau(p)\rhd\eta(p)}} a_\eta B^{\tau(p)}_{\eta(p)} \right) D^-_{\tau(p)} = 0,$$

due to the second order derivatives of f in a term in D_q^- that does not depend on ℓ. The second equation is generically fulfilled, only if

$$\sum_{\tau\in\mathcal{L}_1} \left(a_\tau + \sum_{\substack{\eta\in\mathcal{L}_2 \\ \tau(p)\rhd\eta(p)}} a_\eta B^{\tau(p)}_{\eta(p)} \right) = 0.$$

As the $B^{\tau(p)}_{\eta(p)}$ are rational expressions in $a_{\sigma_1}, \dots, a_{\sigma_n}$ with denominator that generically does not equal a_η, this can only hold true if

$$a_\tau + \sum_{\substack{\eta\in\mathcal{L}_2 \\ \tau(p)\rhd\eta(p)}} a_\eta B^{\tau(p)}_{\eta(p)} = 0$$

for all $\tau \in \mathcal{L}_1$. Using the same argument, this can only be true, if $\mathcal{L}_1 = \emptyset$ which also implies $\mathcal{L}_2 = \emptyset$ as mentioned before.

Assuming this, we obtain from (10.16) that (10.15) generically equals $-\ell$, if and only if

$$-\frac{\sum_{\tau\in\mathcal{L}_0} a_\tau}{\sum_{\sigma\in\Sigma} a_\sigma} = -1. \tag{10.17}$$

For $\mathcal{L}_1 = \mathcal{L}_2 = \emptyset$ we have that $\mathcal{L}_0 = \Sigma \setminus \mathcal{L}_p$. Hence, (10.17) is satisfied if and only if

$$\sum_{\tau\notin\mathcal{L}_p} a_\tau = \sum_{\sigma\in\Sigma} a_\sigma$$

which is equivalent to p being critical. Summarizing, under the given assumptions, condition (T) is fulfilled generically, if and only if p is critical and $q \in B$ for all $q \rhd p$. In all other cases – i.e. when p is non-critical or when p is critical and there exists $q \rhd p$ such that $q \notin B$ – condition (L) is fulfilled.

We start the investigation of branching solutions of x_p for the case that condition (L) is fulfilled. If p is non-critical, the setting is the same as in the first part of *Step* 2. Using Lemma 10.13 we obtain

$$x_p(\lambda) = D_p\lambda + \mathcal{O}\left(|\lambda|^2\right),$$

where

$$D_p = -\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} + \ell}{\sum_{\sigma\in\mathcal{L}_p} a_\sigma}.$$

With a similar argument as before, using Lemma 10.18, we obtain that $D_p = -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}$, and therefore $p \in \mathbb{B}$, generically, if $q \in B$ for all $q \rhd p$, and $p \notin \mathbb{B}$ otherwise.

If p is critical, condition (L) implies the existence of a cell $q \rhd p$ such that $q \notin B$. According to the assumptions, this implies either q critical with $D_q = D_q^-$ or q non-critical and

$$D_q = -\frac{\sum_{\tau\notin\mathcal{L}_q} a_\tau D_{\tau(q)} + \ell}{\sum_{\sigma\in\mathcal{L}_q} a_\sigma} \neq -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}.$$

The situation is as in Lemma 10.14. If

$$\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} + \ell}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}} < 0,$$

the state variable x_p undergoes a saddle node bifurcation as

$$x_p(\lambda) = D_p^\pm \cdot \sqrt{\lambda} + \mathcal{O}\left(|\lambda|\right),$$

where

$$D_p^\pm = \pm\sqrt{-\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} + \ell}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}},$$

which generically does not vanish. If, on the other hand,

$$\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} + \ell}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}} > 0,$$

(10.4) for p has no solutions. In that case, this specific branch for the entire network does not exist. More precisely, the choices of coefficients D_q for $q \rhd p$ were not admissible for the given system parameters. Furthermore, note that

$$\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} + \ell}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}} \neq 0$$

generically, using the condition (L). Hence, there are no other branching solutions. Moreover, the lowest order in λ in solution branches that are possible is 2^{-1}. Hence, we obtain $p \notin \mathbb{B}$ implying $p \notin B$ if p is critical and there is $q \rhd p$ such that $q \notin B$.

Next, we investigate the case that condition (T) is fulfilled. Then p is critical and $q \in B$ for all $q \rhd p$. The setting is as in the second part of *Step* 2. That is, from Lemma 10.15 we obtain

$$x_p(\lambda) = D_p^{\pm}\lambda + \mathcal{O}\left(|\lambda|^2\right)$$

with

$$\begin{aligned} D_p^+ &= -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}, \\ D_p^- &= \frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}\cdot\left(1 + 2\frac{\sum_{\sigma\in\mathcal{L}_p,\tau\notin\mathcal{L}_p} f_{\sigma\tau}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}\right) - \frac{\sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}. \end{aligned} \tag{10.18}$$

Once again, the choice D_p^+ yields $p \in \mathbb{B}$ and D_p^- yields $p \notin \mathbb{B}$.

Step 4. We may now combine *Steps* 2 and 3 in order to inductively add cells to B. The maximal cells all belong to B. Next, assume p non-maximal, with $q \in B$ for all $q \rhd p$. This, in turn, is true for precisely one choice of coefficients D_q for all $q \rhd p$, that is $D_q = -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}$. If there is one cell above p with a different coefficient, we obtain $p \notin \mathbb{B}$ and, in consequence, also $p \notin B$. Hence, the only case, that can provide the fully synchronous solution is $q \in B$ for all $q \rhd p$. If p is non-critical, this follows immediately from the unique solution with suitable coefficient D_p. If p is critical, this is true for the suitable choice of the coefficient, i.e. $D_p = D_p^+$. In both cases $p \in B$ as well. This inductive process terminates at critical cells p with $q \in B$ for all $q \rhd p$, for which the coefficient $D_p = D_p^-$ has been chosen. Then this inductively filled set of cells has the structure of a block. It remains to be seen, that after this process has stopped, no cells, lower in the network with respect to $\unlhd$, remain in the fully synchronous state.

The remaining cells The remainder of the proof is devoted to the investigation of branching solutions of cells $p \notin B$. This is done via induction over μ_p which describes the maximal number of critical cells along paths from B to p that are not in B.

Step 5. In *Step* 4 we have inductively added cells to B starting at the maximal cells. The process ends at cells such that $p \notin B$ but $q \in B$ for all $q \rhd p$. In particular, this implies p critical and

$$x_p(\lambda) = D_p\lambda + \mathcal{O}\left(|\lambda|^2\right)$$

with $D_p = D_p^-$ as in (10.18). As these cells are critical and all their inputs are from inside B, we obtain $\mu_p = 0$, which matches the square root order of their branching solution.

So far, we have used the results from *Step* 3 only to construct the block B as in *Step* 4. However, the computations of branching solutions therein apply to all cells p such that $q \rhd p$ critical implies $s \in B$ for all $s \rhd q$. In particular, they hold for all $p \notin B$ with $\mu_p = 0$, if p is non-critical, and $\mu_p = 1$, if p is critical – the definition of μ_p implies that $q \rhd p$ critical implies $s \in B$ for all $s \rhd q$ in these cases. In *Step* 3 we compute

$$x_p(\lambda) = \begin{cases} D_p\lambda + \mathcal{O}\left(|\lambda|^2\right) & \text{for } p \text{ non-critical,} \\ D_p^{\pm}\sqrt{\lambda} + \mathcal{O}\left(|\lambda|\right) & \text{for } p \text{ critical,} \end{cases}$$

where

$$D_p = -\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} + \ell}{\sum_{\sigma\in\mathcal{L}_p} a_\sigma} \qquad \left(\neq -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}\right),$$

$$D_p^{\pm} = \pm\sqrt{-\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} + \ell}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}}$$

for such cells – this is only true for admissible choices of D_q for all $q \rhd p$, i.e. choices such that the square roots are real. These coefficients can be computed iteratively for non-critical cells p with the above properties until we reach a critical cell. For a critical cell with these properties we have $\mu_p = 1$ and x_p grows asymptotically as 2^{-1}. Then, every cell lower in the network receives an input of such a critical cell, i.e. of lowest order 2^{-1} in λ. Thus, the proof of *Step* 3 does not apply any longer. Furthermore, $\mu_p \geq 1$ for these cells lower in the network so that the considerations above fully describe the case $\mu_p = 0$ (and additionally the case $\mu_p = 1$ when p is critical).

Step 6. We investigate cells $p \notin B$ with $\mu_p > 0$ inductively. Assume

$$x_q(\lambda) = D_q \lambda^{2^{-\mu_q}} + \mathcal{O}\left(|\lambda|^{2^{-(\mu_q - 1)}}\right) \tag{10.19}$$

with $D_q \neq 0$ for all $q \rhd p$ with $q \notin B$. For $q \trianglerighteq p$ define

$$\xi_q = \max_{s \rhd q} \mu_s$$

– setting $\mu_q = 0$ for $q \in B$ – and

$$Q_q = \{ s \rhd q \mid \mu_s = \xi_q \}$$

(compare to (10.8)). Note that $\mu_q = \xi_q$, if q is non-critical, and $\mu_q = \xi_q + 1$, if q is critical. In particular, *Step* 5 characterizes the bifurcating solutions in all cells $p \notin B$ with $\xi_p = 0$. Furthermore, assume that the coefficients D_q for $q \rhd p$ are chosen according to

$$D_q \qquad \begin{cases} \text{as in } \textit{Step } 5 & \text{for} \quad \xi_q = 0 \\ = -\dfrac{\sum_{\tau(q) \in Q_q} a_\tau D_{\tau(q)}}{\sum_{\sigma \in \mathcal{L}_q} a_\sigma} & \text{for} \quad q \text{ non-critical with } \xi_q > 0 \\ = \pm \sqrt{-\dfrac{\sum_{\tau(q) \in Q_q} a_\tau D_{\tau(q)}}{\sum_{\sigma, \tau \in \mathcal{L}_q} f_{\sigma\tau}}} & \text{for} \quad q \text{ critical with } \xi_q > 0. \end{cases} \tag{10.20}$$

As these rules are not consistent, we assume the choice of coefficients to be done such that square roots are real for all $q \rhd p$, once again. Furthermore, assume

$$\sum_{\substack{\tau \notin \mathcal{L}_q \\ \tau(q) \in Q_q}} a_\tau D_{\tau(q)} \neq \begin{cases} -\ell & \text{for} \quad \xi_q = 0, \\ 0 & \text{for} \quad \xi_q > 0, \end{cases} \tag{10.21}$$

for all $q \rhd p$ such that there exists $s \rhd q$ with $s \notin B$. This assumption guarantees that conditions (L) and (SN) from before are fulfilled for all these cells. In *Step* 3 we have seen that (10.21) generically holds for all $p \in B$ as well. Summarizing, we assume that conditions (L) and (SN) are satisfied for all $q \rhd p$ except for those $q \notin B$ that are critical and $s \in B$ for all $s \rhd q$ – i.e. the ones where the inductive process of adding cells to B from *Step* 4 stops. For these condition (T) is satisfied. In particular, this implies $D_q \neq 0$ for all $q \rhd p$ generically.

The case p critical with $\mu_p = 1$, i.e. $\xi_p = 0$, is as in *Step* 5. We obtain

$$x_p(\lambda) = D_p^{\pm} \lambda^{2^{-1}} + \mathcal{O}(|\lambda|)$$

with

$$D_p^{\pm} = \pm \sqrt{-\frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau D_{\tau(p)} + \ell}{\sum_{\sigma, \tau \in \mathcal{L}_p} f_{\sigma\tau}}},$$

which matches the assumptions.

The remaining cells $p \notin B$ with $\mu_p > 0$ all satisfy $\xi_p > 0$. By definition, $\mu_q = \xi_p$ for all $q \in Q_p$. If $q \in Q_p$ is non-critical, $s \in Q_q$ implies $\mu_s = \xi_q = \mu_q = \xi_p$. Hence, $s \in Q_p$ by transitivity of $\trianglelefteq$. If

$q \in Q_p$ is critical on the other hand, $\mu_s = \xi_q = \mu_q - 1 = \xi_p - 1$ for all $s \in Q_q$, which implies $s \notin Q_p$. Summarizing, we see $Q_q \subset Q_p$ for $q \rhd p$ non-critical and $Q_q \cap Q_p = \emptyset$ for q critical. Repeating this consideration and inductively applying assumption (10.20), this implies that the coefficients D_q for all $q \in Q_p$ are determined by those for $\overline{q} \in Q_p$ critical. That is, we have

$$D_{\overline{q}}^{\pm} \in \begin{cases} \left\{ \pm\sqrt{-\frac{\sum_{\tau \notin \mathcal{L}_q} a_\tau D_{\tau(q)} + \ell}{\sum_{\sigma,\tau \in \mathcal{L}_q} f_{\sigma\tau}}} \right\}, & \text{if} \quad \mu_q = \mu_p = 1, \\ \left\{ \pm\sqrt{-\frac{\sum_{\tau(q) \in Q_q} a_\tau D_{\tau(q)}}{\sum_{\sigma,\tau \in \mathcal{L}_q} f_{\sigma\tau}}} \right\}, & \text{if} \quad \mu_q = \mu_p > 1, \end{cases}$$

if $q \in Q_p$ is critical and

$$D_q = \sum_{\overline{q} \in Q_p \text{ critical}} A_{\overline{q}}^q D_{\overline{q}},$$

if $q \in Q_p$ is non-critical. The coefficients $A_{\overline{q}}^q$ are inductively computed using assumption (10.20) on non-critical cells with $\xi_q > 0$ starting from the critical cells $\overline{q} \in Q_p$. They may be zero, but $D_q \neq 0$ by assumption.

We may split $Q_p = Q_1 \sqcup Q_2$ into non-critical and critical cells respectively and obtain

$$\sum_{\substack{\tau \notin \mathcal{L}_q: \\ \tau(p) \in Q_p}} a_\tau D_{\tau(p)} = \sum_{\tau:\, \tau(p) \in Q_1} a_\tau \sum_{\overline{q} \in Q_p \text{ critical}} A_{\overline{q}}^{\tau(p)} D_{\overline{q}} + \sum_{\tau:\, \tau(p) \in Q_2} a_\tau D_{\tau(p)}.$$

Due to the inductive computation of the coefficients $A_{\overline{q}}^{\tau(p)}$, which contain fractions of system parameters a_σ, this sum generically only equals 0, if both summands vanish simultaneously. The coefficients a_τ for $\tau \in \mathcal{L}_2 = \{\tau \notin \mathcal{L}_p \mid \tau(p) \in Q_2\}$ offer enough freedom to show that this generically implies $D_{\tau(p)} = 0$ for all $\tau \in \mathcal{L}_2$ or $\mathcal{L}_2 = \emptyset$. Both cannot hold true, due to the induction assumption. Hence, the sum generically does not vanish and condition (SN) is satisfied. In particular, this proves (10.21) holds true for p as well.

Next, we compute the branching solutions for cell p under the induction assumptions (10.19), (10.20) and (10.21). If p is non-critical, Lemma 10.13 shows that (10.4) for p has the unique branching steady states

$$x_p(\lambda) = -\frac{\sum_{\tau:\, \tau(p) \in Q_p} a_\tau D_{\tau(p)}}{\sum_{\sigma \in \mathcal{L}_p} a_\sigma} \cdot \lambda^{2^{-\xi_p}} + \mathcal{O}\left(|\lambda|^{2^{-(\xi_p - 1)}}\right).$$

As $\xi_p = \mu_p$ for p non-critical, this proves the steady states for cell p branch as in (10.19) with coefficients as in (10.20).

Consider p critical. As we have investigated the case $\mu_p = 1$ already, we may assume $\mu_p > 1$ and therefore $\xi_p > 0$. The setting is as in Lemma 10.16. If

$$\frac{\sum_{\tau:\, \tau(p) \in Q_p} a_\tau D_{\tau(p)}}{\sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau}} < 0,$$

the state variable x_p undergoes a saddle node bifurcation as

$$x_p(\lambda) = D_p^{\pm} \lambda^{2^{-(\xi_p+1)}} + \mathcal{O}\left(|\lambda|^{2^{-\xi_p}}\right)$$

with

$$D_p^{\pm} = \pm\sqrt{-\frac{\sum_{\tau:\, \tau(p) \in Q_p} a_\tau D_{\tau(p)}}{\sum_{\sigma,\tau \in \mathcal{L}_p} f_{\sigma\tau}}},$$

which generically does not vanish. As $\mu_p = \xi_p + 1$, this proves the steady states for cell p branch as in (10.19) with coefficients as in (10.20).

If, on the other hand,

$$\frac{\sum_{\tau:\, \tau(p)\in Q_p} a_\tau D_{\tau(p)}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}} > 0,$$

(10.4) for p has no solutions. In this case the choices for coefficients D_q for $q \triangleright p$ were not admissible for the given system parameters. Furthermore, note that

$$\frac{\sum_{\tau:\, \tau(p)\in Q_p} a_\tau D_{\tau(p)}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}} \neq 0$$

generically, using condition (SN). Hence, there are no other possible branching solutions.

Step 7. *Steps* 5 and 6 serve as an inductive proof for the x_p-coordinates of branching steady states for cells $p \notin B$. Starting with cells such that $\mu_p = 0$, it applies to all p such that there exists $q \triangleright p$ with $q \notin B$. We see that all these cells bifurcate as in (10.19)

$$x_p(\lambda) = D_p\lambda^{2^{-\mu_p}} + \mathcal{O}\left(|\lambda|^{2^{-(\mu_p-1)}}\right)$$

with coefficients as in (10.20). Replacing $\xi_p = \mu_p$, if p is non-critical, and $\xi_p = \mu_p - 1$, if p is critical, yields the coefficients stated in the theorem. In particular, this also proves that $p \notin B$. In turn, this means that $p \in B$ implies $q \in B$ for all $q \triangleright p$, completing the proof that B is a block. If no choice of coefficients D_q for $q \triangleright p$ according to the inductive rules (10.20) is possible, the specific branch does not exist. In particular, the choice of the block B is not admissible. This completes the proof for the branching steady states as stated in the proposition.

□

Correspondingly, we obtain an analogous result for $\lambda < 0$ as a corollary of Theorem 10.20.

Theorem 10.21 (Subcritical branches)**.** *Under the bifurcation assumption* (B) *with non-critical maximal cells, generically,*[2] *all bifurcating branches of steady states for $\lambda < 0$ are as follows: There is a maximal block $B \subset C$ such that the state variable x_p for each $p \in B$ remains in the fully synchronous state $x_p(\lambda) = X(\lambda)$ as in* (10.9)*. Maximality means that there is no block with the same properties containing B.*

The state variables of all $p \notin B$ bifurcate as

$$x_p(\lambda) = D_p \cdot (-\lambda)^{2^{-\mu_p}} + \mathcal{O}\left(|\lambda|^{2^{-(\mu_p-1)}}\right),$$

where $D_p \neq 0$ and

$$\mu_p = \max_{\overline{p}\in B}\ \max_{\omega\in\Omega_{\overline{p},p}} \#\{q\in\omega \mid q \textit{ critical}, q\notin B\} - 1$$

is the maximal number of critical cells $q \notin B$ along paths from any cell in B to p.

The branch exists, if and only if

$$\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} - \ell}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}} < 0, \qquad \frac{\sum_{\tau:\, \tau(p)\in Q_p} a_\tau D_{\tau(p)}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}} < 0$$

for all critical cells p with $\mu_p = 1$ or $\mu_p > 1$ respectively. Therein, Q_p is defined analogous to (10.8) *with ξ_p replaced by μ_p.*

[2] i.e. for an open and dense subset of system parameters $\{a_\sigma, f_{\sigma\tau}, f_{\sigma\lambda}, \ell, f_{\lambda\lambda} \mid \sigma,\tau\in\Sigma\}$

The D_p are chosen according to the inductive rules:

D_p	*Case*
$-\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} - \ell}{\sum_{\sigma\in\mathcal{L}_p} a_\sigma}$	p *non-critical*, $\mu_p = 0$
$-\frac{\sum_{\tau\colon \tau(p)\in Q_p} a_\tau D_{\tau(p)}}{\sum_{\sigma\in\mathcal{L}_p} a_\sigma}$	p *non-critical*, $\mu_p > 0$
$-\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}\cdot\left(1+2\frac{\sum_{\sigma\in\mathcal{L}_p,\tau\notin\mathcal{L}_p} f_{\sigma\tau}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}\right)+\frac{\sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}$	p *critical*, $\mu_p = 0$
$\pm\sqrt{-\frac{\sum_{\tau\notin\mathcal{L}_p} a_\tau D_{\tau(p)} - \ell}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}}$	p *critical*, $\mu_p = 1$
$\pm\sqrt{-\frac{\sum_{\tau\colon \tau(p)\in Q_p} a_\tau D_{\tau(p)}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}}$	p *critical*, $\mu_p > 1$

Proof. Substituting the parameter $\mu = -\lambda$ and the system parameters $\iota = -\ell, \theta_{\sigma\lambda} = -f_{\sigma\lambda}$, the governing function Taylor expands as

$$f(x_{\sigma_1(p)},\ldots,x_{\sigma_n(p)},\lambda) = \sum_{\sigma\in\Sigma} a_\sigma x_{\sigma(p)} + \iota\mu + \sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}x_{\sigma(p)}x_{\tau(p)} + \sum_{\sigma\in\Sigma}\theta_{\sigma\lambda}\mu x_{\sigma(p)} + f_{\lambda\lambda}\mu^2 + \text{h.o.t.},$$

as in (10.4). The result follows immediately from Theorem 10.20 for $\mu > 0$. □

The branching solutions in Theorems 10.20 and 10.21 contain those that are described in Section 6 in Soares [99] for layered feedforward networks as a special case. Therein, generically all non-maximal cells are critical if the maximal cells are non-critical. This generalization is due to the fact that the class of feedforward networks satisfying the equivalent definitions in Theorem 8.35 contains layered feedforward networks (compare to Remark 8.37).

Remark 10.22. We refer to branches for $\lambda > 0$ as *supercritical* and for $\lambda < 0$ as *subcritical*. △

Remark 10.23. The proofs of Theorems 10.20 and 10.21 provide a constructive method to determine all branching solutions for the bifurcation problem (B). The first step is to determine all blocks $B \subset C$ such that $p \notin B$ with $q \in B$ for all $q \rhd p$ implies p is critical. A solution branch is computed as x_p staying in the fully synchronous state for all $p \in B$ and the states of the remaining cells being determined by the number of critical cells in between p and B where the lowest order coefficients D_p are chosen according to the inductive rules in the statements. Note that the asymptotic order can also be determined inductively by setting $\mu_p = 0$ for all $p \in B$ and

$$\mu_p = \begin{cases} \max_{q\rhd p}\mu_q & \text{for} \quad p \text{ non-critical,} \\ \max_{q\rhd p}\mu_q + 1 & \text{for} \quad p \text{ critical.}\end{cases}$$

This will be made more precise in Proposition 10.27 below. △

Remark 10.24. Note that the conditions determining the existence of branching solutions in Theorems 10.20 and 10.21 depend only on the system parameters – the partial derivatives of f. This implies the existence of different branches in different regions of system parameter space which may also vary according to the direction of branching – super- or subcritical. △

Remark 10.25. Finally, we remark that the possible choices of a block $B \subset C$ of cells that remain in the fully synchronous state (as in Remark 10.23) are independent of super- or subcritical branching. Choosing the same block in Theorems 10.20 and 10.21, we see that within B the solutions exist for both signs of λ. Similarly, when $\mu_p = 0$ there is always a pair of coefficients with opposite signs for super- and subcritical branches. Furthermore, investigating (10.4) from that perspective, we see that the higher order terms match as well and the solutions in these cells are truly transcritical. However, cells lower in the network may still prohibit the existence of a particular solution branch for one or both signs of λ. $\triangle$

10.3 The amplification effect

We conclude this chapter by making the notion of the *amplification effect* in generic steady state bifurcations of feedforward networks precise. That is, in both major cases observed in Sections 10.1 and 10.2 the branching solution in a given cell has a steeper slope the lower the cell is in the network with respect to $\trianglelefteq$. The discussion in this section essentially consists of a summary of Theorems 10.8, 10.20 and 10.21 similar to that in the beginning of this chapter while taking a slightly different perspective.

A large part of Sections 10.1 and 10.2 is devoted to not only characterize generic steady state bifurcations of a feedforward network but also to provide formulas to precisely compute the lowest order coefficients depending on the Taylor coefficients of the response function. As we indicated before, in bifurcation analysis one is often not interested in the explicit expression describing how to compute the branching state state for every parameter value but rather in the qualitative behavior. To that end, we consider the asymptotics of the state variable for each cell separately in a generic steady state bifurcation. As existing branches may differ over regions in system parameter space, these asymptotics may vary accordingly. However, whenever two branches differ only by the explicit expression for its lowest order coefficients the asymptotics of each cell are the same for both branches. In particular, this is the case on open subsets of the parameter space. That is why we choose to encode the qualitative bifurcation information of each individual cell in its the asymptotics of its state variable. We introduce some notation to make these ideas more precise.

Consider a generic branch of steady states for a parameter dependent feedforward network vector field $(x_p(\lambda))_{p\in C} \subset \bigoplus_{p\in C} \mathbb{R}$ for small $\lambda > 0, \lambda < 0$ or $|\lambda|$ as provided in Theorems 10.8 and 10.20 or Theorem 10.21. Recall from Theorems 10.20 and 10.21 that in the case of non-critical maximal cells there is a block $B \subset C$ of cells that remain in the fully synchronous state

$$X(\lambda) = D\lambda + \mathcal{O}\left(|\lambda|^2\right) \tag{10.22}$$

as in Lemma 10.17. The lowest order in λ of the branching states of the remaining cells is determined by the constant μ_p as

$$x_p(\lambda) = D_p\lambda^{2^{-\mu_p}} + \mathcal{O}\left(|\lambda|^{2^{-(\mu_p-1)}}\right) \tag{10.23}$$

(if the branch is subcritical reverse the sign of λ). The definition of μ_p can readily be extended to reflect the lowest order in λ also for cells $p \in B$ by defining $\mu_p = 0$ in that case. Furthermore, recall from Theorem 10.8 that the state of each cell $p \in C$ is given by

$$x_p(\lambda) = D_p \cdot \lambda^{2^{-1}} + \mathcal{O}(|\lambda|) \tag{10.24}$$

(once again reverse the sign of λ if the branch is subcritical) in the case of critical maximal cells. Hence, we may extend the definition of μ_p to reflect the lowest order in λ for all cells when the maximal cells are critical by $\mu_p = 1$ for all $p \in C$ in that case. Summarizing, we see that

$$x_p(\lambda) = D_p\lambda^{2^{-\mu_p}} + \mathcal{O}\left(|\lambda|^{2^{-(\mu_p-1)}}\right)$$

with some suitable coefficient $D_p \neq 0$ for all $p \in C$ in any generic branch of steady states of a feedforward network. This motivates the following definition.

Definition 10.26. *Let $(x_p(\lambda))_{p\in C} \subset \bigoplus_{p\in C}\mathbb{R}$ be a branch of steady states for a parameter dependent feedforward network vector field for small $\lambda > 0, \lambda < 0$ or $|\lambda|$ as provided in Theorem 10.8, Theorem 10.20, and Theorem 10.21. Then we say that a cell $p \in C$* has the square root order μ_p in λ *or that it* grows asymptotically as $\lambda^{2^{-\mu_p}}$ *where μ_p is as defined above.*

The main qualitative characteristic of steady state bifurcations in feedforward networks – namely the *amplification effect* – can now comfortably be described using Definition 10.26. The following proposition shows that only critical cells have an amplifying effect as they have a higher square root order than the cells they receive an input from. This can almost directly be seen from the original definition of μ_p in Theorems 10.20 and 10.21 (compare also to Remark 10.23).

Proposition 10.27. *Let $(x_p(\lambda))_{p\in C} \subset \bigoplus_{p\in C}\mathbb{R}$ be a branch of steady states for a parameter dependent feedforward network vector field for small $\lambda > 0, \lambda < 0$ or $|\lambda|$ as provided in Theorem 10.8, Theorem 10.20, and Theorem 10.21 with cell-by-cell asymptotics $x_p \sim \lambda^{2^{-\mu_p}}$ for all $p \in C$. Then the square root orders are amplified by critical cells. That is, for p maximal we have $\mu_p = 1$ or $\mu_p = 0$ when the maximal cells are critical or non-critical respectively. For p non-maximal we obtain*

$$\mu_p = \begin{cases} \max_{q\rhd p}\mu_q & \textit{for} \quad p \textit{ non-critical}, \\ \max_{q\rhd p}\mu_q & \textit{for} \quad p \textit{ critical and } q\in B \textit{ for all } q\rhd p, \\ \max_{q\rhd p}\mu_q + 1 & \textit{for} \quad p \textit{ critical and there exists } q\rhd p \textit{ such that } q\notin B. \end{cases} \tag{10.25}$$

In particular, the square root orders of the cells are partially ordered with respect to $\trianglelefteq$, i.e.

$$p \trianglelefteq q \qquad \textit{implies} \qquad \mu_p \geq \mu_q. \tag{10.26}$$

Hence, if $p \trianglelefteq q$, then x_p has at least the same asymptotic order as x_q.

Proof. In the case of critical maximal cells, only the maximal cells are critical. Furthermore all cells have square root order 1, i.e. $x_p \sim \sqrt{\lambda}$ as seen from (10.24). Hence, nothing is to be shown.

If, on the other hand, the maximal cells are non-critical, we obtain a block of cells $B \subset C$ such that all cells in B have square root order 0: $x_p \sim \lambda$ – or $\mu_p = 0$ – for all $p \in B$ as seen in (10.22). In particular, B contains all maximal cells. Furthermore, if p is critical and $q \in B$ for all $q \rhd p$ then also $x_p \sim \lambda$. This matches the definition of μ_p in Theorems 10.20 and 10.21. The state variable of the remaining cells $p \notin B$ bifurcate as in (10.23) with square root order

$$\mu_p = \max_{\overline{p}\in B}\max_{\omega\in\Omega_{\overline{p},p}} \#\left\{q\in\omega \mid q \text{ critical}, q\notin B\right\} - 1.$$

We immediately see that this definition implies

$$\mu_p = \begin{cases} \max_{q\rhd p}\mu_q & \text{for} \quad p \text{ non-critical}, \\ \max_{q\rhd p}\mu_q + 1 & \text{for} \quad p \text{ critical}. \end{cases}$$

for all $p \notin B$ where additionally there exists a cell $q \rhd p$ such that $q \notin B$ if p is critical. This completes the proof of (10.25). From this (10.26) follows directly. □

In particular, Proposition 10.27 shows that an actual amplification can only be observed when the maximal cells are non-critical. Then only the critical cells outside of the block of cells that remain in the fully synchronous state have an amplifying effect.

Chapter 11

Steady state bifurcations in feedforward networks with high-dimensional internal dynamics

This chapter investigates generic 1-parameter steady state bifurcations in feedforward networks with high-dimensional internal dynamics. In Theorems 10.8, 10.20 and 10.21 we give a precise characterization of all generic branching solutions in the one-dimensional case. Theorem 7.17 then implies that the generic steady state bifurcations in the case of high-dimensional dynamics are the same qualitatively – when we restrict ourselves to fundamental networks. In particular, coordinate free bifurcation diagrams in both cases look the same. As we laid out in Section 10.3 the most important feature of the bifurcations in feedforward networks is the amplification effect. This is a qualitative property of each solution branch. However, it is not qualitative in the sense that it remains under arbitrary coordinate transformations but rather a qualitative property of each individual cell. Hence, one should be able to observe it in a generic steady state bifurcation in all coordinates that reflect individual cells. In this chapter we show that feedforward structure can be used to observe the amplification effect also in generic steady state bifurcations when the internal dynamics is high-dimensional. The most important tools are the partial order on the cells and the fact that the admissible maps are upper triangular – or more precisely respect the partial order. As the center manifold reduction for fundamental networks respects monoid symmetry, this in particular holds for the maps involved in the procedure that allows to translate a generic branch in the one-dimensional case into a generic branch in the high-dimensional case in Section 7.2.4. As equivariance is equivalent to admissibility, this has the convenient effect that whenever we compute some property of the state variable x_p of cell p it only depends on the state variables of cells $q \trianglerighteq p$. This observation proves to be powerful enough to translate the amplification effect into the high-dimensional case without having to determine center manifolds explicitly. As the results in Chapter 7 only apply to fundamental networks, we restrict ourselves to fundamental feedforward networks and note that the bifurcation results Theorems 10.8, 10.20 and 10.21 apply to this case nonetheless. We use the same notational conventions as in Chapter 7. In particular, we refer to the two cases as **1D** and **DD** respectively. Furthermore, when distinction is necessary we denote the internal and total phase space by $V = \mathbb{R}$ and $\mathbf{V}^1 = \bigoplus_{\sigma\in\Sigma} \mathbb{R}$ in the case **1D** and $V = W \cong \mathbb{R}^d$ and $\mathbf{V}^{\mathsf{D}} = \bigoplus_{\sigma\in\Sigma} W$ in the case **DD** with respective coordinates $v = (v_\sigma)_{\sigma\in\Sigma} \in \mathbf{V}^1$ and $\omega = (w_\sigma)_{\sigma\in\Sigma} \in \mathbf{V}^{\mathsf{D}}$. If an observation holds true in both cases we keep using $x = (x_\sigma)_{\sigma\in\Sigma} \in \bigoplus_{\sigma\in\Sigma} V$. Once again, we drop the convention of denoting coordinates in the phase spaces of a fundamental network by capital letters. The results in this chapter are from Section 4 in our preprint [83].

The amplification effect in Proposition 10.27 is described in terms of the square root order of individual cells in a given branch of bifurcating steady state solutions. In order to be able to characterize similar phenomena in feedforward networks with high-dimensional internal dynamics we generalize Definition 10.26 to that case. Even more so the following definition is independent of the feedforward network structure. It could also have been formulated for branching solutions in arbitrary networks.

Definition 11.1. *Let $(x_\sigma(\lambda))_{\sigma\in\Sigma} \subset \bigoplus_{\sigma\in\Sigma} V$, for small $\lambda > 0, \lambda < 0$ or $|\lambda|$ small, be a branch of steady states for a parameter dependent fundamental network vector field, i.e.*

$$\Gamma_f((x_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) = 0. \tag{11.1}$$

We say that a cell $\sigma \in \Sigma$ has the square root order $\xi_\sigma \geq 0$ in λ *or that it* grows asymptotically as $\lambda^{2^{-\xi_\sigma}}$ *if*

$$\begin{cases} \|x_\sigma(\lambda)\| = d_\sigma \cdot \lambda^{2^{-\xi_\sigma}} + \mathcal{O}\left(|\lambda|^{2^{-(\xi_\sigma+1)}}\right) & \text{with } d_\sigma \neq 0 \quad \text{for} \quad \xi_\sigma \geq 1, \\ \|x_\sigma(\lambda)\| = \mathcal{O}\left(|\lambda|\right) & \text{for} \quad \xi_\sigma = 0. \end{cases}$$

Therein $\|\cdot\|$ denotes the euclidean norm on V. If the branch only exists for $\lambda < 0$ replace λ by $-\lambda$. We denote this situation by $x_\sigma \sim \lambda^{2^{-\xi_\sigma}}$.

Remark 11.2. Note that $\xi_\sigma = 0$ summarizes all cases in which x_σ does not actually grow with any square root order. That is, it grows at least linearly, up to lowest order in λ. △

Remark 11.3. The situation $x_\sigma \sim \lambda^{2^{-\xi_\sigma}}$ with $\xi_\sigma > 0$ is equivalent to

$$x_\sigma(\lambda) = \lambda^{2^{-\mu_\sigma}} \cdot \vartheta_\sigma + R_\sigma(\lambda),$$

where $\vartheta_\sigma \in V \setminus \{0\}$ suitable and $R_\sigma \colon \mathbb{R} \to V$ – restricted to the suitable neighborhood of $\lambda_0 = 0$ – such that $\|R_\tau(\lambda)\| = \mathcal{O}\left(|\lambda|^{2^{-(\mu_\tau-1)}}\right)$. △

A crucial part of this chapter contains technical investigations of the maps that are needed to define the center manifold reduction as in Section 3.5 and their interaction with branching steady state solutions. In particular, we will make heavy use of the technicalities proved in Section 3.5.3. We begin by briefly reformulating the setting and recalling some necessary notation from Sections 3.5 and 7.2.4. In the remainder of this chapter we consider fundamental feedforward networks. In particular $\trianglelefteq$ is a partial order. We investigate steady state solutions of $\Gamma_f \colon \bigoplus_{\sigma\in\Sigma} V \times \mathbb{R} \to \bigoplus_{\sigma\in\Sigma} V$, which is an admissible vector field for a feedforward fundamental network as in (10.1) that depends on a real parameter $\lambda \in \mathbb{R}$ close to the bifurcation point $(x_0, \lambda_0) = (0, 0)$. That is, we are interested in solutions to

$$\Gamma_f(x, \lambda) = 0,$$

close to $(0, 0)$, where

$$D_x\Gamma_f(0, 0)$$

is non-invertible. Once again, this singular linear admissible map induces a splitting

$$\bigoplus_{\sigma\in\Sigma} V = \ker_0(D_x\Gamma_f(0,0)) \oplus \operatorname{im}_0(D_x\Gamma_f(0,0))$$

into generalized kernel and reduced image with corresponding projections

$$P^c \colon \bigoplus_{\sigma\in\Sigma} V \to \ker_0(D_x\Gamma_f(0,0))$$
$$P^h \colon \bigoplus_{\sigma\in\Sigma} V \to \operatorname{im}_0(D_x\Gamma_f(0,0))$$

that are equivariant with respect to the right regular representation. As stated before, for technical reasons, we focus on the extended system as in (3.20)

$$\underline{\dot{x}} = \begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} \Gamma_f(x, \lambda) \\ 0 \end{pmatrix} = \underline{\Gamma}_f(\underline{x}) \tag{11.2}$$

on $\bigoplus_{\sigma\in\Sigma} V \times \mathbb{R}$. Solutions of (11.2) are in one-to-one correspondence with those of (11.1). Furthermore, the extended system is equivariant with respect to the extended right regular representation $\sigma \mapsto \underline{A}_\sigma$ where

$$\underline{A}_\sigma \colon \underline{x} = (x, \lambda) \mapsto (A_\sigma x, \lambda).$$

Under the bifurcation assumption the linearization of the extended system induces a splitting of $\bigoplus_{\sigma\in\Sigma} V \times \mathbb{R}$ into subrepresentations

$$\begin{aligned}\underline{\mathcal{X}}^h &= \mathrm{im}_0(D_x\Gamma_f(0,0)) \times \{0\},\\ \underline{\mathcal{X}}^c &= \ker_0(D_x\Gamma_f(0,0)) \times \{0\} \oplus \left\langle (x^h, 1) \right\rangle,\end{aligned} \tag{11.3}$$

where $x^h \in \mathrm{im}_0(D_x\Gamma_f(0,0))$ suitable, as in (3.21). These subspaces are generalized kernel and reduced image of $D\underline{\Gamma}_f(0) = D_{(x,\lambda)}\underline{\Gamma}_f(0,0)$ respectively. The corresponding equivariant projections are denoted by

$$\begin{aligned}\underline{P}^c &\colon \bigoplus_{\sigma\in\Sigma} V \times \mathbb{R} \to \underline{\mathcal{X}}^c\\ \underline{P}^h &\colon \bigoplus_{\sigma\in\Sigma} V \times \mathbb{R} \to \underline{\mathcal{X}}^h.\end{aligned} \tag{11.4}$$

In particular, every element $\underline{x}^c \in \underline{\mathcal{X}}^c$ can be represented as

$$\underline{x}^c = (x^c, 0) + \lambda(x^h, 1),$$

where $x^c \in \ker_0(D_x\Gamma_f(0,0))$ and $\lambda \in \mathbb{R}$. Furthermore, there is an isomorphism of monoid representations $P' \colon \underline{\mathcal{X}}^c \to \ker_0(D_x\Gamma_f(0,0)) \times \mathbb{R}$ given by

$$P'(\underline{x}^c) = (P^c(x^c + \lambda x^h), \lambda) = (x^c, \lambda) \tag{11.5}$$

with inverse

$$Q'(x^c, \lambda) = (x^c, 0) + \lambda(x^h, 1) = \underline{x}^c. \tag{11.6}$$

Then the extended system admits the local center manifold

$$M^c = \left\{ \left(\mathbb{1}_{\underline{\mathcal{X}}^c} + \underline{\psi}\right)(\underline{x}^c) \mid \underline{x}^c \in \underline{\mathcal{X}}^c \right\} = \left\{ \left(\mathbb{1}_{\underline{\mathcal{X}}^c} + \underline{\psi}\right)(Q'(x^c, \lambda)) \mid (x^c, \lambda) \in \ker_0(D_x\Gamma_f(0,0)) \times \mathbb{R} \right\}, \tag{11.7}$$

which is represented as a graph over either $\underline{\mathcal{X}}^c$ or equivalently over $\ker_0(D_x\Gamma_f(0,0)) \times \mathbb{R}$ by the equivariant map $\underline{\psi}$.

The center manifold contains all solutions of the extended system with initial conditions close to the bifurcation point whose $\underline{\mathcal{X}}^h$-component is bounded. In particular, for any branch of steady states of the bifurcation problem we have $((x_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) \subset M^c$ – we will not include the restriction of λ to a suitable neighborhood of 0 in the remainder but assume it to be present nevertheless. Furthermore, the dynamics restricted to the center manifold is bijectively conjugate to dynamics on the center subspace $\underline{\mathcal{X}}^c$ or on $\ker_0(D_x\Gamma_f(0,0)) \times \mathbb{R}$ (compare to Theorem 3.32). Hence, any branch can uniquely be represented in coordinates of these subspaces using the projections $\underline{P}^c$ or $P = P' \circ \underline{P}^c$ such that

$$\begin{aligned}\underline{P}^c\left((x_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda\right) &\subset \underline{\mathcal{X}}^c,\\ P'\left(\underline{P}^c\left((x_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda\right)\right) &\subset \ker_0(D_x\Gamma_f(0,0)) \times \mathbb{R}\end{aligned}$$

(compare to Section 7.2.4). Recall from (3.22) and Lemmas 3.36 to 3.39 that all maps required to switch between the three representation of a branch of steady states – i.e. $\underline{P}^c, P', Q'$ and $\underline{\psi}$ leave the

λ-coordinate unchanged (the third representation is the original representation $((x_\sigma(\lambda))_{\sigma\in\Sigma},\lambda)$). Hence, we see that the parameter of each representation remains the same and we may write

$$\begin{aligned} ((y_\sigma(\lambda))_{\sigma\in\Sigma},\lambda) &= \underline{P}^c\left((x_\sigma(\lambda))_{\sigma\in\Sigma},\lambda\right) \subset \underline{\mathcal{X}}^c, \\ ((z_\sigma(\lambda))_{\sigma\in\Sigma},\lambda) &= P'\left(\underline{P}^c\left((x_\sigma(\lambda))_{\sigma\in\Sigma},\lambda\right)\right) \subset \ker_0(D_x\Gamma_f(0,0)) \times \mathbb{R}. \end{aligned} \tag{11.8}$$

Note that the second representation of (11.8) is of particular interest as it describes the branch of steady states in the corresponding reduced bifurcation problem on $\ker_0(D_x\Gamma_f(0,0)) \times \mathbb{R}$ (see Theorem 3.32).

We investigate to what extent cell-by-cell asymptotics are respected when switching between the different representations. In particular, we answer the following question: Under the assumption of square root orders of individual cells that are partially ordered as in Proposition 10.27 for feedforward networks – more precisely as in (10.26) – in either one of the representations, can we recover the square root order of each cell in any of the other representations? As the distinction of cells is reflected in the choice of coordinates, this is done by analyzing the projection operators in the original coordinates.

Lemma 11.4. *(i) Let $((x_\sigma(\lambda))_{\sigma\in\Sigma},\lambda) \subset M^c$ be a branch of steady states of* (11.2). *Assume cell-by-cell asymptotics $x_\sigma \sim \lambda^{2^{-\mu_\sigma}}$ that are partially ordered as in* (10.26), *i.e. $\sigma \trianglerighteq \tau$ implies $\mu_\sigma \leq \mu_\tau$. Then the representation in the generalized kernel of the extended system*

$$((y_\sigma(\lambda))_{\sigma\in\Sigma},\lambda) = \underline{P}^c\left((x_\sigma(\lambda))_{\sigma\in\Sigma},\lambda\right) \subset \underline{\mathcal{X}}^c$$

exhibits the same cell-by-cell asymptotics, i.e.

$$y_\sigma \sim \lambda^{2^{-\mu_\sigma}} \quad \textit{for all} \quad \sigma \in \Sigma.$$

(ii) Let $((y_\sigma(\lambda))_{\sigma\in\Sigma},\lambda) \subset \underline{\mathcal{X}}^c$ be the representation of a branch of steady states of (11.2) *in the generalized kernel of the extended system. Assume cell-by-cell asymptotics $y_\sigma \sim \lambda^{2^{-\mu_\sigma}}$ that are partially ordered as in* (10.26), *i.e. $\sigma \trianglerighteq \tau$ implies $\mu_\sigma \leq \mu_\tau$. Then the representation in the full space*

$$((x_\sigma(\lambda))_{\sigma\in\Sigma},\lambda) = \left(\mathbb{1}_{\underline{\mathcal{X}}^c} + \underline{\psi}\right)\left((y_\sigma(\lambda))_{\sigma\in\Sigma},\lambda\right) \subset M^c$$

exhibits the same cell-by-cell asymptotics, i.e.

$$x_\sigma \sim \lambda^{2^{-\mu_\sigma}} \quad \textit{for all} \quad \sigma \in \Sigma.$$

Proof. The central observation to prove both parts of the lemma is the fact that the maps $\underline{P}^c|_{M^c} : M^c \to \underline{\mathcal{X}}^c$ and $\left(\mathbb{1}_{\underline{\mathcal{X}}^c} + \underline{\psi}\right) : \underline{\mathcal{X}}^c \to M^c$ are mutually inverse bijections between the center manifold M^c and the generalized kernel of the extended system $\underline{\mathcal{X}}^c$. The properties of these maps are well-understood from Lemmas 3.36 to 3.38. We know that

$$\underline{P}^c(x,\lambda) = \left(\underline{P}^c_x(x,\lambda),\lambda\right),$$

where $\underline{P}^c_x$ can be interpreted as a parameter dependent linear map on $\bigoplus_{\sigma\in\Sigma} V$ that is equivariant with respect to the non-extended monoid representation, i.e. $\underline{P}^c_x(A_\sigma x,\lambda) = A_\sigma \underline{P}^c_x(x,\lambda)$ for all $x \in \bigoplus_{\sigma\in\Sigma} V, \lambda \in \mathbb{R}$ and $\sigma \in \Sigma$. Due to the equivalence of equivariance and admissibility, it is therefore a parameter-dependent linear admissible map of the fundamental network vector field. In particular, it respects the partial order $\trianglelefteq$

$$\left(\underline{P}^c_x(x,\lambda)\right)_\tau = \ell_\tau\left((x_\sigma \mid \sigma \trianglerighteq \tau),\lambda\right), \tag{11.9}$$

where ℓ_τ is a linear map that depends only on the entries x_σ for $\sigma \trianglerighteq \tau$ and the parameter λ. The parameter-dependence is linear as well.

A similar observation holds for the other direction. Denote elements of the generalized kernel of the extended system by $(y, \lambda) = ((y_\sigma)_{\sigma\in\Sigma}, \lambda) \in \underline{\mathcal{X}}^c \subset \bigoplus_{\sigma\in\Sigma} V \times \mathbb{R}$. Then we know $\underline{\psi}(y, \lambda) = (\underline{\psi}_x(y, \lambda), 0)$, where $\underline{\psi}_x$ is a parameter-dependent admissible map of the fundamental network. That is

$$\left(\underline{\psi}_x(y, \lambda)\right)_\tau = q_\tau\left((y_\sigma \mid \sigma \trianglerighteq \tau), \lambda\right). \tag{11.10}$$

As $\underline{\psi}(0, 0) = 0$ and $D\underline{\psi}(0, 0) = 0$, we also obtain $q_\tau(0, 0) = 0$ and $dq_\tau(0, 0) = 0$.

We may now prove both cases separately beginning with (ii). Let $((y_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) \subset \underline{\mathcal{X}}^c$ be the representation of a branch of steady states in the generalized kernel of the extended system. Assume cell-by-cell asymptotics $y_\sigma \sim \lambda^{2^{-\mu_\sigma}}$ that satisfy the ordering assumption (10.26), i.e. $\sigma \trianglerighteq \tau$ implies $\mu_\sigma \leq \mu_\tau$. First, we fix a cell $\tau \in \Sigma$ with $\mu_\tau \geq 1$. By definition $y_\tau \sim \lambda^{2^{-\mu_\tau}}$ is equivalent to

$$y_\tau(\lambda) = \lambda^{2^{-\mu_\tau}} \cdot \vartheta_\tau + R_\tau(\lambda),$$

where $\vartheta_\tau \in V \setminus \{0\}$ suitable and $\|R_\tau(\lambda)\| = \mathcal{O}\left(|\lambda|^{2^{-(\mu_\tau-1)}}\right)$. Combining the partial order of the cell-by-cell asymptotics with the form of the map whose graph is the center manifold (11.10), we compute

$$\left(\underline{\psi}_x\left((y_\sigma(\lambda)_{\sigma\in\Sigma}, \lambda)\right)\right)_\tau = q_\tau\left((y_\sigma(\lambda) \mid \sigma \trianglerighteq \tau), \lambda\right) = \lambda^{2^{-(\mu_\tau-1)}} \cdot \eta_\tau + \rho_\tau(\lambda), \tag{11.11}$$

where $\eta_\tau \in V$ suitable and $\|\rho_\tau(\lambda)\| = \mathcal{O}\left(|\lambda|^{2^{-(\mu_\tau-1)}}\right)$.

As we obtain the representation of the branch on the center manifold $((x_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda)$ from

$$((x_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) = \left(\mathbb{1}_{\underline{\mathcal{X}}^c} + \underline{\psi}\right)\left((y_\sigma(\lambda)_{\sigma\in\Sigma}, \lambda)\right) = \left((y_\sigma(\lambda))_{\sigma\in\Sigma} + \underline{\psi}_x((y_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda), \lambda\right),$$

we directly see

$$x_\tau(\lambda) = \lambda^{2^{-\mu_\tau}} \cdot \vartheta_\tau + R_\tau(\lambda) + \lambda^{2^{-(\mu_\tau-1)}} \cdot \eta_\tau + \rho_\tau(\lambda) = \lambda^{2^{-\mu_\sigma}} \cdot \vartheta_\tau + \varkappa_\tau(\lambda),$$

where $\|\varkappa_\tau(\lambda)\| = \mathcal{O}\left(|\lambda|^{2^{-(\mu_\tau-1)}}\right)$. Hence, $x_\tau \sim \lambda^{2^{-\mu_\tau}}$, since $\vartheta_\tau \neq 0$.

On the other hand, consider a cell $\tau \in \Sigma$ with $\mu_\tau = 0$. By the partial order of the square root orders (10.26) we also obtain $\mu_\sigma = 0$ for all $\sigma \trianglerighteq \tau$. By definition this is equivalent to $\|y_\sigma(\lambda)\| = \mathcal{O}(|\lambda|)$ for all $\sigma \trianglerighteq \tau$ including τ. Thus, using (11.10) as before, we compute

$$\|x_\tau(\lambda)\| = \|y_\tau(\lambda) + q_\tau\left((y_\sigma(\lambda) \mid \sigma \trianglerighteq \tau), \lambda\right)\| = \mathcal{O}(|\lambda|),$$

which proves $x_\tau \sim \lambda$ and, therefore, completes the proof for (ii).

Next we turn to the proof of the first statement (i). This is slightly more complicated as we have to make sure that the projection onto the generalized kernel of the extended system does not 'loose' information about the asymptotics of a given cell. Assume that $((x_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) \subset M^c$ is a branch of steady states whose cell-by-cell asymptotics satisfy the ordering assumption (10.26). Then, fixing a cell $\tau \in \Sigma$ with $\mu_\tau \geq 1$, we obtain

$$x_\tau(\lambda) = \lambda^{2^{-\mu_\tau}} \cdot \vartheta_\tau + R_\tau(\lambda),$$

with $\vartheta_\tau \in V \setminus \{0\}$ suitable and $\|R_\tau(\lambda)\| = \mathcal{O}\left(|\lambda|^{2^{-(\mu_\tau-1)}}\right)$. Combining the partial order of the cell-by-cell asymptotics with the specific form of the projection onto $\underline{\mathcal{X}}^c$ (11.9), we obtain

$$\left(\underline{P}^c_x\left((x_\sigma(\lambda)_{\sigma\in\Sigma}, \lambda)\right)\right)_\tau = \ell_\tau\left((x_\sigma(\lambda) \mid \sigma \trianglerighteq \tau), \lambda\right) = \lambda^{2^{-\mu_\tau}} \cdot \eta_\tau + \rho_\tau(\lambda), \tag{11.12}$$

where $\eta_\tau \in V$ suitable and $\|\rho_\tau(\lambda)\| = \mathcal{O}\left(|\lambda|^{2^{-(\mu_\tau-1)}}\right)$. The representation of the branch in the generalized kernel of the extended system $((y_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda)$ is computed as

$$((y_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) = \underline{P}^c\left((x_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda\right) = \left(\underline{P}^c_x\left((x_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda\right), \lambda\right).$$

Thus, we see that $y_\tau \sim \lambda^{2-\mu_\tau}$, if $\eta_\tau \neq 0$.[1]

However, as $((y_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) \subset \underline{\mathcal{X}}^c$ is the representation of a branch of steady states in the generalized kernel of the extended system, we may apply the results from (ii) to it and receive its corresponding representation on the center manifold $((\tilde{x}_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) \subset M^c$ with the same cell-by-cell asymptotics. As in the proof of part (ii), we compute its x-component to be

$$\tilde{x}_\tau(\lambda) = y_\tau(\lambda) + \underline{\psi}_x((y_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) = \lambda^{2-\mu_\tau} \cdot \eta_\tau + \varkappa_\tau(\lambda), \tag{11.13}$$

where $\eta_\tau \in V$ is as in (11.12) and $\|\varkappa_\tau(\lambda)\| = \mathcal{O}\left(|\lambda|^{2-(\mu_\tau-1)}\right)$. Furthermore, $\underline{P}^c|_{M^c}$ and $\left(\mathbb{1}_{\underline{\mathcal{X}}^c} + \underline{\psi}\right)$ are mutually inverse bijections between M^c and $\underline{\mathcal{X}}^c$. Hence,

$$\begin{aligned}((x_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) &= \left(\left(\mathbb{1}_{\underline{\mathcal{X}}^c} + \underline{\psi}\right) \circ \underline{P}^c{}_{|M^c}\right)((x_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda)\\ &= \left(\mathbb{1}_{\underline{\mathcal{X}}^c} + \underline{\psi}\right)((y_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda)\\ &= ((\tilde{x}_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda)\,.\end{aligned}$$

Using (11.13), we obtain

$$x_\tau(\lambda) = \lambda^{2-\mu_\tau} \cdot \vartheta_\tau + R_\tau(\lambda) = \lambda^{2-\mu_\tau} \cdot \eta_\tau + \varkappa_\tau(\lambda) = \tilde{x}_\tau(\lambda).$$

In particular, $\eta_\tau = \vartheta_\tau \neq 0$, which proves $y_\tau \sim \lambda^{2-\mu_\sigma}$.

On the other hand, $\mu_\tau = 0$ implies $\mu_\sigma = 0$ and therefore $\|x_\tau(\lambda)\| = \mathcal{O}(|\lambda|)$ for all $\sigma \trianglerighteq \tau$ (which includes τ). Thus we compute

$$\|y_\tau(\lambda)\| = \|\ell_\tau\left((x_\sigma(\lambda) \mid \sigma \trianglerighteq \tau), \lambda\right)\| = \mathcal{O}(|\lambda|),$$

which shows $y_\tau \sim \lambda$, completing the proof of (i). ☐

Remark 11.5. The proof for Lemma 11.4 allows for a slightly more precise statement when relaxing the ordering assumption on the cell-by-cell asymptotics (10.26). That is, for every cell τ for which the ordering assumption is fulfilled, $\mu_\sigma \leq \mu_\tau$ for all $\sigma \trianglerighteq \tau$, we obtain that the τ-component exhibits the same asymptotics in the full representation of the solution branch and in its representation in the generalized kernel of the extended system, i.e. $x_\tau \sim \lambda^{2-\mu_\tau}$ and $y_\tau \sim \lambda^{2-\mu_\tau}$. This does not require the square root orders of *all* cells to be ordered. △

In the previous lemma we have shown that, under the ordering assumption (10.26), cell-by-cell asymptotics agree in the representation of a steady state branch on the center manifold with those in the representation in the generalized kernel of the extended system. A similar result holds true when comparing representations in $\underline{\mathcal{X}}^c$ and in $\ker_0(D_x\Gamma_f(0,0)) \times \mathbb{R}$ even without the ordering assumption, i.e. $y_\tau \sim z_\tau$ with y_τ and z_τ as in (11.8). This follows from the properties of the mutually inverse isomorphisms P' and Q'.

Lemma 11.6. *(i) Let $((y_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) \subset \underline{\mathcal{X}}^c$ be the representation of a branch of steady states of* (11.2) *in the generalized kernel of the extended system. Assume cell-by-cell asymptotics $y_\sigma \sim \lambda^{2-\mu_\sigma}$. Then the representation*

$$((z_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) = \underline{P}'((y_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) \subset \ker_0(D_x\Gamma_f(0,0)) \times \mathbb{R}$$

exhibits the same cell-by-cell asymptotics, i.e.

$$z_\sigma \sim \lambda^{2-\mu_\sigma} \quad \textit{for all} \quad \sigma \in \Sigma.$$

[1]The opposite case $\eta_\tau = 0$ is what we refer to as ‘loosing’ information about the asymptotics in cell τ.

(ii) Let $((z_\sigma(\lambda))_{\sigma\in\Sigma},\lambda) \subset \ker_0(D_x\Gamma_f(0,0))\times\mathbb{R}$ be the representation of a branch of steady states of (11.2) *in $\ker_0(D_x\Gamma_f(0,0))$. Assume cell-by-cell asymptotics $z_\sigma \sim \lambda^{2^{-\mu_\sigma}}$. Then the representation in the generalized kernel of the extended system*

$$((y_\sigma(\lambda))_{\sigma\in\Sigma},\lambda) = Q'\left((z_\sigma(\lambda))_{\sigma\in\Sigma},\lambda\right) \subset \underline{\mathcal{X}}^c$$

exhibits the same cell-by-cell asymptotics, i.e.

$$y_\sigma \sim \lambda^{2^{-\mu_\sigma}} \quad \textit{for all} \quad \sigma\in\Sigma.$$

Proof. We prove (i) and (ii) similarly to the proof of Lemma 11.4. The maps $P'\colon (y,\lambda)\mapsto (P^c(y),\lambda)$ as in (11.5) and $Q'\colon (z,\lambda)\mapsto (z+\lambda x^h,\lambda)$ as in (11.6) form equivariant isomorphisms between $\underline{\mathcal{X}}^c$ and $\ker_0(D_x\Gamma_f(0,0))\times\mathbb{R}$. In particular, due to Lemma 3.36 we know that P^c is equivariant with respect to the non-extended representation $\sigma\mapsto A_\sigma$. Hence, it is a linear, admissible map for the fundamental network and we obtain

$$\left(P^c\left((y_\sigma)_{\sigma\in\Sigma}\right)\right)_\tau = \ell_\tau\left(y_\sigma \mid \sigma \trianglerighteq \tau\right).$$

Let $((y_\sigma(\lambda))_{\sigma\in\Sigma},\lambda)\subset\underline{\mathcal{X}}^c$ be the representation of a branch of steady states of (11.2) in the generalized kernel of the extended system and assume cell-by-cell asymptotics $y_\sigma\sim\lambda^{2^{-\mu_\sigma}}$. For a cell τ with $\mu_\tau\geq 1$, this is equivalent to

$$y_\tau(\lambda) = \lambda^{2^{-\mu_\tau}}\cdot\vartheta_\tau + R_\tau(\lambda),$$

where $\vartheta_\tau\in V\setminus\{0\}$ and $\|R_\tau(\lambda)\| = \mathcal{O}\left(|\lambda|^{2^{-(\mu_\tau-1)}}\right)$. As $((z_\sigma(\lambda))_{\sigma\in\Sigma},\lambda) = \left(P^c\left((y_\sigma(\lambda))_{\sigma\in\Sigma}\right),\lambda\right)$, we compute

$$z_\tau(\lambda) = \ell_\tau\left(y_\sigma(\lambda)\mid\sigma\trianglerighteq\tau\right) = \lambda^{2^{-\xi_\tau}}\cdot\eta_\tau + \rho_\tau(\lambda),$$

where $\eta_\tau\in V, \xi_\tau = \max_{\sigma\trianglerighteq\tau}\mu_\sigma$ and $\|\rho_\tau(\lambda)\| = \mathcal{O}\left(|\lambda|^{2^{-(\xi_\tau-1)}}\right)$. Furthermore, note that

$$Q'\left((z_\sigma(\lambda))_{\sigma\in\Sigma},\lambda\right) = \left((z_\sigma(\lambda))_{\sigma\in\Sigma} + \lambda\cdot(x^h_\sigma)_{\sigma\in\Sigma},\lambda\right).$$

Therein

$$z_\tau(\lambda)+\lambda\cdot x^h_\tau = \lambda^{2^{-\xi_\tau}}\cdot\eta_\tau + \lambda\cdot x^h_\tau + \rho_\tau(\lambda) = \lambda^{2^{-\xi_\tau}}\cdot\eta_\tau + \varkappa_\tau(\lambda),$$

where $\|\varkappa_\tau(\lambda)\| = \mathcal{O}\left(|\lambda|^{2^{-(\xi_\tau-1)}}\right)$, since $\mu_\tau\geq 1$. As P' and Q' are mutually inverse, we obtain

$$y_\tau(\lambda) = \lambda^{2^{-\mu_\tau}}\cdot\vartheta_\tau + R_\tau(\lambda) = z_\tau(\lambda)+\lambda\cdot x^h_\tau = \lambda^{2^{-\xi_\tau}}\cdot\eta_\tau + \varkappa_\tau(\lambda).$$

Thus, $\eta_\tau = \vartheta_\tau \neq 0$ and $\xi_\tau = \mu_\tau$ so that $z_\tau\sim\lambda^{2^{-\mu_\tau}}$. If on the other hand $\mu_\tau = 0$, we obtain $\mu_\sigma = 0$ and therefore $y_\sigma\sim\lambda$ for all $\sigma\trianglerighteq\tau$, including τ. We compute

$$\|z_\tau(\lambda)\| = \|\ell_\tau\left((y_\sigma(\lambda)\mid\sigma\trianglerighteq\tau),\lambda\right)\| = \mathcal{O}(|\lambda|),$$

which shows $z_\tau\sim\lambda$, completing the proof of (i).

Conversely, consider the representation of a branch of steady states in $\ker_0(D_x\Gamma_f(0,0))\times\mathbb{R}$ given by $((z_\sigma(\lambda))_{\sigma\in\Sigma},\lambda)$ with cell-by-cell asymptotics $z_\sigma\sim\lambda^{2^{-\mu_\sigma}}$. Fix a cell $\tau\in\Sigma$ with $\mu_\tau\geq 1$ first. Then

$$z_\tau(\lambda) = \lambda^{2^{-\mu_\tau}}\cdot\vartheta_\tau + R_\tau(\lambda)$$

with $\vartheta_\tau\in V\setminus\{0\}$ and $\|R_\tau(\lambda)\| = \mathcal{O}\left(|\lambda|^{2^{-\mu_\tau-1}}\right)$. As before, we compute the representation in the generalized kernel of the extended system to be

$$y_\tau(\lambda) = z_\tau(\lambda)+\lambda\cdot x^h_\tau = \lambda^{2^{-\mu_\tau}}\vartheta_\tau + \lambda\cdot x^h_\tau + R_\tau(\lambda).$$

As $\vartheta_\tau\neq 0$, this implies $y_\tau\sim\lambda^{2^{-\mu_\tau}}$. On the other hand, assume $\mu_\tau = 0$, which implies $\|z_\tau(\lambda)\| = \mathcal{O}(|\lambda|)$ as before. In particular, we compute

$$\|y_\tau(\lambda)\| = \left\|z_\tau(\lambda)+\lambda\cdot x^h_\tau\right\| = \mathcal{O}(|\lambda|),$$

which shows $y_\tau\sim\lambda$ and therefore completes the proof of (ii). □

Corollary 11.7. *In feedforward fundamental networks, the cell-by-cell asymptotics of the full representation of branches of steady states of* (11.2)*, agree with those of the representation in* $\ker_0(D_x\Gamma_f(0,0)) \times \mathbb{R}$*, if they respect the ordering relation* (10.26)*. That is the cell-by-cell asymptotics are invariant under the bijective conjugation* $P = P' \circ \underline{P}^c \colon M^c \to \mathcal{X}^c \times \mathbb{R}$.

Remark 11.8. Similar to Remark 11.5 a slightly more precise statement relaxing the ordering assumption on the cell-by-cell asymptotics (10.26) is possible. For every cell τ for which the ordering assumption is fulfilled, $\mu_\sigma \leq \mu_\tau$ for all $\sigma \trianglerighteq \tau$, we obtain that the τ-component exhibits the same asymptotics in every representation of the solution branch. Here we do not require the square root orders of *all* cells to be ordered. △

These technical results put us in the position to prove the main theorems of the investigation of 1-parameter steady state bifurcations in fundamental feedforward networks with high-dimensional internal dynamics. For bifurcation problems in which the generalized kernel at the bifurcation point fulfills an additional condition, cell-by-cell asymptotics turn out to be the same as in the case **1D**. This follows almost directly from Corollary 11.7. In an arbitrary generic bifurcation problem only a slightly weaker statement holds. That is, the amplification effect as in Proposition 10.27 is the same in both cases. The precise square root order of each cell, however, is not necessarily the same in both cases. In order to properly formulate the results we make use of the tensor notation introduced in Section 7.1. In particular, we use indices and superscripts 1 and D with objects that have been defined before to indicate one-dimensional or d-dimensional internal dynamics respectively without explicitly defining them. For example Γ^1_ϕ is a fundamental network vector field with internal phase space $\mathbb{R}$ and response function ϕ, while Γ^{D}_f is an admissible vector field of the same network with internal phase space W and response function f. Furthermore, recall that the bifurcation setting in the form of assumption (B) in Chapter 10 is formulated independent of the dimension of the internal phase space, even if we only classify bifurcations in the case **1D** there. Hence, we refer to this assumption in the case **DD** as well.

Theorem 11.9. *Consider a feedforward fundamental network with one-dimensional internal dynamics and a generic* 1*-parameter family of admissible vector fields satisfying the bifurcation assumption (B) from Chapter 10 with the absolutely indecomposable subrepresentation* $Y \subset \mathbf{V}^1 = \bigoplus_{\sigma\in\Sigma} \mathbb{R}$ *as the generalized kernel at the bifurcation point. Denote the set of all* 1*-parameter families of admissible vector fields satisfying the bifurcation assumption (B) from Chapter 10 with* Y *as the generalized kernel at the bifurcation point by*

$$\mathcal{F} = \left\{ \Gamma^1_\phi \colon \mathbf{V}^1 \times \mathbb{R} \to \mathbf{V}^1 \mid \ker_0(D_x\Gamma^1_\phi(0,0)) = Y \right\}.$$

Furthermore, consider a generic 1*-parameter family* $\Gamma^{\mathsf{D}}_f \colon \mathbf{V}^{\mathsf{D}} \times \mathbb{R} \to \mathbf{V}^{\mathsf{D}}$ *of admissible vector fields on the same network with internal phase space* $W \cong \mathbb{R}^d$ *with* $d > 1$ *satisfying the bifurcation assumption (B) from Chapter 10 and the additional condition*

$$\ker_0(D_\omega\Gamma^{\mathsf{D}}_f(0,0)) = Y \otimes \langle w \rangle \tag{11.14}$$

on the kernel in tensor notation, where $w \in W \setminus \{0\}$ *(see Theorem 7.12).*[2] *Then there is a generic* $\Gamma^1_\phi \in \mathcal{F}$ *such that* Γ^{D}_f *exhibits the same pattern of local branches of steady states with the same cell-by-cell asymptotics.*

More precisely, the branches of steady states of Γ^1_ϕ *are known from Theorems 10.8, 10.20 and 10.21. Denote them by* $(v_\sigma(\lambda))_{\sigma\in\Sigma} \subset \mathbf{V}^1$*. In particular, each cell* $\sigma \in \Sigma$ *is of square root order* μ_σ*, i.e.* $v_\sigma \sim \lambda^{2^{-\mu_\sigma}}$ *with* μ_σ *as in Definition 10.26. Then each branch of steady states* $(w_\sigma(\lambda))_{\sigma\in\Sigma} \subset \mathbf{V}^{\mathsf{D}}$ *of* Γ^{D}_f *uniquely corresponds to a branch* $(v_\sigma(\lambda))_{\sigma\in\Sigma}$ *of* Γ^1_ϕ *and square root orders are the same in both branches, i.e.* $w_\sigma \sim \lambda^{2^{-\mu_\sigma}}$ *for all* $\sigma \in \Sigma$.

[2] In the original coordinates this means for any basis $(b^1_\sigma)_{\sigma\in\Sigma}, \dots, (b^k_\sigma)_{\sigma\in\Sigma}$ of Y the elements $(b^i_\sigma w)_{\sigma\in\Sigma}$ span $\ker_0(\Gamma^{\mathsf{D}}_f(0,0))$.

Proof. Let $\Gamma_f^{\mathsf{D}} \colon \mathbf{V}^{\mathsf{D}} \times \mathbb{R} \to W$ be a generic 1-parameter family of admissible vector fields with $\Gamma_f^{\mathsf{D}}(0,0) = 0$ and

$$\ker_0(D_\omega \Gamma_f^{\mathsf{D}}(0,0)) = Y \otimes \langle w \rangle,$$

where $Y \subset \mathbf{V}^1$ is an absolutely indecomposable subrepresentation. Due to the center manifold reduction, the dynamics of Γ_f^{D} on its center manifold M^c is bijectively conjugate to that of a generic vector field

$$F\colon \ker_0(D_\omega \Gamma_f^{\mathsf{D}}(0,0)) \times \mathbb{R} \to \ker_0(D_\omega \Gamma_f^{\mathsf{D}}(0,0)).$$

In particular, all bifurcating branches of steady states of Γ_f^{D} can uniquely be represented as branches of steady states of F in $\ker_0(D_\omega \Gamma_f^{\mathsf{D}}(0,0)) \times \mathbb{R}$. As the center manifold reduction preserves symmetry, the generic steady state bifurcations of the reduced system are entirely classified by Σ-equivariance.

The same observation holds true for a generic 1-parameter steady state bifurcation in the case **1D**. Let $\Gamma_\phi^1 \colon \mathbf{V}^1 \times \mathbb{R} \to \mathbb{R}$ be a generic 1-parameter family of admissible vector fields satisfying the bifurcation assumption (B) from Chapter 10. Its dynamics on the center manifold is bijectively conjugate to that of a generic equivariant system

$$G\colon \ker_0(D_v \Gamma_\phi^1(0,0)) \times \mathbb{R} \to \ker_0(D_v \Gamma_\phi^1(0,0))$$

In particular, if $\Gamma_\phi^1 \in \mathcal{F}$ the generalized kernel satisfies

$$\ker_0(D_v \Gamma_\phi^1(0,0)) = Y$$

Once again, dynamics on Y is classified by Σ-symmetry. As a result, there is a generic choice $\Gamma_\phi^1 \in \mathcal{F}$ such that

$$F(y \otimes w, \lambda) = G(y, \lambda) \otimes w \tag{11.15}$$

(recall that every element in $Y \otimes \langle w \rangle$ can be represented as a pure tensor $y \otimes w$ from the proof of Lemma 7.10).

On the other hand, from Theorems 10.8, 10.20 and 10.21, we know all branching solutions for a generic Γ_ϕ^1. In particular, for any branch $(x_\sigma(\lambda))_{\sigma\in\Sigma} \subset \mathbf{V}^1$ we know that $x_\sigma \sim \lambda^{2^{-\mu_\sigma}}$, where the μ_σ are as in Definition 10.26 and satisfy the ordering (10.26). Corollary 11.7 shows that the representation $(y_\sigma(\lambda))_{\sigma\in\Sigma}$ on Y has the same cell-by-cell asymptotics $y_\sigma \sim \lambda^{2^{-\mu_\sigma}}$. For this representation we have $G((y_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) = 0$ for all λ with small absolute value.

Applying (11.15) this directly implies

$$F((y_\sigma(\lambda))_{\sigma\in\Sigma} \otimes w, \lambda) = G((y_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) \otimes w = 0.$$

Hence, we obtain a branch of steady states of the generic equivariant vector field F on $Y \otimes \langle w \rangle$ by attaching the vector $w \in W \setminus \{0\}$ to each coordinate of the branch $y(\lambda)$:

$$(y_\sigma(\lambda))_{\sigma\in\Sigma} \otimes w \subset \ker_0(D_\omega \Gamma_f^{\mathsf{D}}(0,0)).$$

Furthermore, as the dynamics on $Y \times \mathbb{R}$ and $Y \otimes \langle w \rangle \times \mathbb{R}$ are determined by symmetry, we obtain all branches of steady states of F in this way (the representation on $\mathbf{V}^{\mathsf{D}}$ via $\{A_\sigma \otimes \mathbb{1}_W\}_{\sigma\in\Sigma}$ is trivial in the $\langle w \rangle$-component). In the original coordinates this branch is denoted by

$$(z_\sigma(\lambda))_{\sigma\in\Sigma} = (y_\sigma(\lambda) \cdot w)_{\sigma\in\Sigma}$$

(see (7.2)). In particular $z_\sigma \sim \lambda^{2^{-\mu_\sigma}}$. Due to the center manifold reduction, this branch uniquely corresponds to a branch $(w_\sigma(\lambda))_{\sigma\in\Sigma}$ of the full system governed by Γ_f^{D}. As $(y_\sigma(\lambda))_{\sigma\in\Sigma}$ satisfies the ordering (10.26) on its cell-by-cell asymptotics, Corollary 11.7 implies $w_\sigma \sim \lambda^{2^{-\mu_\sigma}}$. Note that all steps in this proof require the application of bijective maps. Hence, we indeed have a one-to-one correspondence between branches in the **1D**-case and branches in the **DD**-case. $\square$

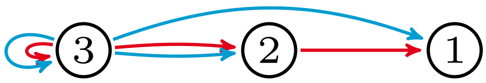


Figure 11.1: A 3-cell homogeneous feedforward chain.

Remark 11.10. Note that condition (11.14) imposes a restriction of generality. In general, due to Theorem 7.12, the generalized kernel is isomorphic – as a subrepresentation – but not equal to $Y \otimes \langle w \rangle$. $\triangle$

Example 11.1. We illustrate the previous remark by recalling Example 1.1 from the introduction. The network in Figure 11.1 is the network in Figure 1.3 after its input maps have been completed to a monoid. It is easy to see, that it is also a feedforward fundamental network. Similar to before, we observe that the linearization of a 1-parameter family of admissible vector fields satisfying the bifurcation assumption (B) from Chapter 10 is of the form

$$L = \begin{pmatrix} A & B & C \\ 0 & A & B+C \\ 0 & 0 & A+B+C \end{pmatrix},$$

where $A, B, C \in \mathfrak{gl}(V)$. Under the assumption of non-critical maximal cells, we have a (generically simple) eigenvalue 0 of A – i.e. $A = 0$ in the case **1D** – and we compute

$$\ker_0^1(L^1) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{B} \\ 0 \end{pmatrix} \right\rangle, \qquad \ker_0^{\mathsf{D}}(L^{\mathsf{D}}) = \left\langle \begin{pmatrix} Y \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Y' \\ Y \\ 0 \end{pmatrix} \right\rangle,$$

where $Y, Y' \in W$ are such that $AY = 0$ and $AY' = (\mathbb{1}_V - B)Y$. Hence, we see

$$\ker_0^{\mathsf{D}}(L^{\mathsf{D}}) \neq \left\langle \begin{pmatrix} Y \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{B}Y \\ 0 \end{pmatrix} \right\rangle \cong \ker_0^1(L^1) \otimes \langle Y \rangle.$$

Nevertheless, it can easily be verified that

$$\begin{aligned} A^{\mathsf{D}}_{\sigma_2} \begin{pmatrix} Y' \\ Y \\ 0 \end{pmatrix} &= \begin{pmatrix} Y \\ 0 \\ 0 \end{pmatrix}, \qquad & A^{\mathsf{D}}_{\sigma_3} \begin{pmatrix} Y' \\ Y \\ 0 \end{pmatrix} &= 0 \\ A^{\mathsf{D}}_{\sigma_2} \begin{pmatrix} 0 \\ \frac{1}{B}Y \\ 0 \end{pmatrix} &= \begin{pmatrix} \frac{1}{B}Y \\ 0 \\ 0 \end{pmatrix}, \qquad & A^{\mathsf{D}}_{\sigma_3} \begin{pmatrix} 0 \\ \frac{1}{B}Y \\ 0 \end{pmatrix} &= 0. \end{aligned}$$

In particular the actions of σ_2 and σ_3 on $(Y', Y, 0)^T$ and on $(0, \frac{1}{B}Y, 0)^T$ are conjugate. Hence, we have indeed

$$\ker_0^{\mathsf{D}}(L^{\mathsf{D}}) \cong \left\langle \begin{pmatrix} Y \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{B}Y \\ 0 \end{pmatrix} \right\rangle \cong \ker_0^1(L^1) \otimes \langle Y \rangle.$$

$\triangle$

Theorem 11.9 describes generic steady state bifurcations in a **DD**-feedforward network only in the special case that the generalized kernel of the vector field is essentially equal to one that also occurs generically in the **1D**-case. For a generic feedforward fundamental network with high-dimensional internal dynamics the result is slightly weaker. Here we only obtain the same cell asymptotics as in the **1D**-case for cells $\sigma \in \Sigma$ for which $\mu_\sigma > \mu_\sigma$. In particular, this is the case for cells that are critical in the **1D**-setting. In cells that are non-critical the square root order in the case **DD** is less than or equal to that in the case **1D**. However, as amplification in the case **1D** is only visible in the critical cells – μ_σ increases only in critical cells (see Proposition 10.27) –, this can be interpreted as the same amplifying effect in steady state bifurcations in networks with high-dimensional internal dynamics. In order to make the notion of genericity precise, we need to make the following remark first.

Remark 11.11. Consider a generic 1-parameter family of admissible vector fields for a feedforward fundamental network $\Gamma_f^{\mathsf{D}}\colon \mathbf{V}^{\mathsf{D}} \times \mathbb{R} \to \mathbf{V}^{\mathsf{D}}$ satisfying the bifurcation assumption (B) from Chapter 10 in the case **DD**. As generic steady state bifurcations occur along absolutely indecomposable subrepresentations, we know that $\ker_0(D_\omega \Gamma_f^{\mathsf{D}}(0,0))$ is absolutely indecomposable. Furthermore, due to Theorem 7.12, there is a decomposition into indecomposable subrepresentations

$$\mathbf{V}^1 = Y_1 \oplus \cdots \oplus Y_s$$

such that

$$\ker_0(D_\omega \Gamma_f^{\mathsf{D}}(0,0)) \cong Y_j \otimes \langle w \rangle \cong Y_j$$

for some $1 \leq j \leq s$. Furthermore, any subrepresentation Y_i can occur in a generic 1-parameter steady state bifurcation in the case **1D** – according to Theorems 9.1 and 9.2 the subrepresentations are all absolutely indecomposable and in one-to-one correspondence with the eigenvalues of a generic linear admissible map. That is, there is a generic 1-parameter family of admissible vector fields for the same fundamental feedforward network $\Gamma_\phi^1\colon \mathbf{V}^1 \times \mathbb{R} \to \mathbf{V}^1$ satisfying the bifurcation assumption (B) from Chapter 10 in the case **1D** such that

$$\ker_0(D_v \Gamma_\phi^1(0,0)) \cong Y_j \cong \ker_0(D_\omega \Gamma_f^{\mathsf{D}}(0,0)),$$

where the isomorphisms are isomorphisms of subrepresentations. △

Theorem 11.12. *Consider a feedforward fundamental network with internal phase space $W \cong \mathbb{R}^d$ with $d > 1$. Let $\Gamma_f^{\mathsf{D}}\colon \mathbf{V}^{\mathsf{D}} \times \mathbb{R} \to \mathbf{V}^{\mathsf{D}}$ be a generic 1-parameter family of admissible vector fields satisfying the bifurcation assumption (B) from Chapter 10. Then Γ_f^{D} exhibits the same amplification effect in its branching steady state solutions as a generic 1-parameter family of admissible vector fields for the same network with one-dimensional internal dynamics $\Gamma_\phi^1\colon \mathbf{V}^1 \times \mathbb{R} \to \mathbf{V}^1$.*

That is, any branch $(w_\sigma(\lambda))_{\sigma\in\Sigma}$ of steady states of Γ_f^{D} uniquely corresponds to a branch $(v_\sigma(\lambda))_{\sigma\in\Sigma}$ of Γ_ϕ^1. These are known from Theorems 10.8, 10.20 and 10.21. In particular, each cell $\sigma \in \Sigma$ is of square root order μ_σ, i.e. $v_\sigma \sim \lambda^{2^{-\mu_\sigma}}$ with μ_σ as in Definition 10.26. Then

$$w_\sigma \sim \lambda^{2^{-\xi_\sigma}} \quad \textit{for all} \quad \sigma \in \Sigma,$$

where

$$\xi_\sigma \begin{cases} = \mu_\sigma & \textit{for} \quad \sigma \textit{ critical}, \\ \in \{0, \ldots, \mu_\sigma\} & \textit{for} \quad \sigma \textit{ non-critical}. \end{cases}$$

Here, criticality is to be understood with respect to Γ_ϕ^1 – the definition of criticality depends on a non-invertible linearization of a 1-parameter family of admissible vector fields.

Proof. The general idea of the proof is similar to that of Theorem 11.9. Therefore, we omit some of the details and focus on the difficulties that arise when condition (11.14) is violated. Once again, dynamics restricted to the center manifold of a generic parameter dependent system $\Gamma_f^{\mathsf{D}}\colon \mathbf{V}^{\mathsf{D}} \times \mathbb{R} \to \mathbf{V}^{\mathsf{D}}$ satisfying the bifurcation assumption (B) from Chapter 10 is bijectively conjugate to that given by a generic equivariant vector field

$$F\colon \ker_0(D_\omega \Gamma_f^{\mathsf{D}}(0,0)) \times \mathbb{R} \to \ker_0(D_\omega \Gamma_f^{\mathsf{D}}(0,0)).$$

Due to Theorem 7.12, we know that

$$\mathbf{V}^{\mathsf{D}} \supset \ker_0(D_\omega \Gamma_f^{\mathsf{D}}(0,0)) \cong Y \otimes \langle w \rangle, \tag{11.16}$$

where $w \in W \setminus \{0\}$ and $Y \subset \mathbf{V}^1$ is an absolutely indecomposable subrepresentation. Remark 11.11 implies, that there is a generic 1-parameter family of admissible vector fields $\Gamma_\phi^1 \colon \mathbf{V}^1 \times \mathbb{R} \to \mathbf{V}^1$ for the same network with one-dimensional internal dynamics such that

$$\ker_0(D_v \Gamma_\phi^1(0,0)) \cong Y \subset \mathbf{V}^1. \tag{11.17}$$

Generic dynamics – most importantly steady state bifurcations – on Y is uniquely determined by symmetry. On the other hand, the same holds for generic dynamics on $\ker_0(D_v \Gamma_\phi^1(0,0))$. As in the proof of Theorem 11.9, these are also determined by the reduction of the generic system Γ_ϕ^1 to its generalized kernel by the center manifold reduction. In particular, using Theorems 10.8, 10.20 and 10.21 and Corollary 11.7, we obtain a unique representation $(y_\sigma(\lambda))_{\sigma\in\Sigma} \subset \ker_0(D_v \Gamma_\phi^1(0,0))$ of each branch of Γ_ϕ^1 such that $y_\sigma \sim \lambda^{2^{-\mu_\sigma}}$, where μ_σ is defined as in Definition 10.26. In the same manner we obtain all branching steady state solutions of a generic $A_\sigma^1 \otimes \mathbb{1}_W$-equivariant system

$$G \colon \ker_0(D_v \Gamma_\phi^1(0,0)) \otimes \langle w \rangle \times \mathbb{R} \to \ker_0(D_v \Gamma_\phi^1(0,0)) \otimes \langle w \rangle$$

as

$$(y_\sigma(\lambda))_{\sigma\in\Sigma} \otimes w \subset \ker_0(D_v \Gamma_\phi^1(0,0)) \otimes \langle w \rangle.$$

This is due to the obvious observation

$$\ker_0(D_v \Gamma_\phi^1(0,0)) \otimes \langle w \rangle \cong \ker_0(D_v \Gamma_\phi^1(0,0))$$

as representations with respect to $\sigma \mapsto A_\sigma^1 \otimes \mathbb{1}_W$ and $\sigma \mapsto A_\sigma^1$ respectively. In the remainder of this proof it is convenient not to use the tensor notation but to rely on the original coordinates $(y_\sigma w)_{\sigma\in\Sigma} \in \ker_0(D_v \Gamma_\phi^1(0,0)) \otimes \langle w \rangle$. In particular, the generic branch in $\ker_0(D_v \Gamma_\phi^1(0,0)) \otimes \langle w \rangle$ becomes $(y_\sigma(\lambda)w)_{\sigma\in\Sigma}$. It satisfies

$$y_\sigma(\lambda) w \sim \lambda^{2^{-\mu_\sigma}}$$

for all $\sigma \in \Sigma$ as w is fixed.

Using (11.16) and (11.17) we see that there is an isomorphism of subrepresentations

$$\Psi \colon \ker_0(D_\omega \Gamma_f^{\mathsf{D}}(0,0)) \to \ker_0(D_v \Gamma_\phi^1(0,0)) \otimes \langle w \rangle.$$

with inverse Ξ. This isomorphism conjugates generic dynamics on the two spaces as it respects symmetry. In particular all branches of steady states of the generic reduced system F are of the form

$$(z_\sigma(\lambda))_{\sigma\in\Sigma} = \Xi\left((y_\sigma(\lambda))_{\sigma\in\Sigma} w\right) \subset \ker_0(D_\omega \Gamma_f^{\mathsf{D}}(0,0)).$$

Similar to the proof of Theorem 11.9 we have found a relation of the generic branches of steady states in the case **DD** to those in the case **1D**. However, due to the characterization of $(z_\sigma(\lambda))_{\sigma\in\Sigma}$ using the isomorphism Ξ it is not as obvious what the square root orders of each cell are in this representation. Hence, we cannot apply Corollary 11.7 directly to obtain square root orders of the corresponding branches of Γ_f^{D}. Hence, we begin by investigating the effect of the isomorphism Ψ on cell-by-cell square root orders similarly to Lemmas 11.4 and 11.6. To that end, we extend Ψ and Ξ trivially to the full space, i.e. $\Psi, \Xi \colon \mathbf{V}^{\mathsf{D}} \to \mathbf{V}^{\mathsf{D}}$. Hence, both maps are equivariant maps with respect to $\sigma \mapsto A_\sigma^{\mathsf{D}}$ (they are, however, only invertible when restricted to the respective subrepresentations). Due to the equivalence of equivariance and admissibility, they are therefore linear admissible maps for the feedforward fundamental network with internal phase space W. In particular, both maps respect the partial order $\trianglelefteq$ so that we may write

$$(\Psi((Z_\sigma)_{\sigma\in\Sigma}))_\tau \;= \ell_\tau(Z_\sigma \mid \sigma \trianglerighteq \tau) \;= \ell^y_\tau(Z_\sigma \mid \sigma \trianglerighteq \tau)\cdot w, \tag{11.18a}$$

$$(\Xi((Y_\sigma w)_{\sigma\in\Sigma}))_\tau = \mathfrak{l}_\tau(Y_\sigma w \mid \sigma \trianglerighteq \tau) = \sum_{\sigma\trianglerighteq\tau} Y_\sigma \mathfrak{l}^\sigma_\tau(w), \tag{11.18b}$$

where $\ell_\tau\colon W^{\#\{\sigma\trianglerighteq\tau\}} \to W$, $\ell^y_\tau\colon W^{\#\{\sigma\trianglerighteq\tau\}} \to \mathbb{R}$ as well as $\mathfrak{l}_\tau\colon : W^{\#\{\sigma\trianglerighteq\tau\}} \to W$ and $\mathfrak{l}^\sigma_\tau\colon W \to W$ are suitable linear maps.

Similar to the proofs of Lemmas 11.4 and 11.6, we investigate the implications of this structure of the linear maps Ψ and Ξ on the cell-by-cell asymptotics of $(z_\sigma(\lambda))_{\sigma\in\Sigma} = \Xi\left((y_\sigma(\lambda))_{\sigma\in\Sigma}w\right)$. By definition

$$y_\sigma(\lambda) = \begin{cases} \alpha_\sigma\cdot\lambda^{2^{-\mu_\sigma}} + \mathcal{O}\left(|\lambda|^{2^{-(\mu_\sigma-1)}}\right) & \text{if} \quad \mu_\sigma \geq 1,\\ \mathcal{O}(|\lambda|) & \text{if} \quad \mu_\sigma = 0,\end{cases}$$

where $\alpha_\sigma \in \mathbb{R}\setminus\{0\}$. Consider a cell $\tau\in\Sigma$ with $\mu_\tau = 0$ first. By definition $\mu_\sigma = 0$ for all $\sigma \trianglerighteq \tau$. Hence, using (11.18b) we obtain

$$\|z_\tau(\lambda)\| = \left\|\sum_{\sigma\trianglerighteq\tau} y_\sigma(\lambda)\mathfrak{l}^\sigma_\tau(w)\right\| = \mathcal{O}(|\lambda|)$$

and, therefore, $z_\tau \sim \lambda$. In particular τ is of square root order $\iota_\tau = 0$ in that representation.

Next, consider a cell $\tau\in\Sigma$ with $\mu_\tau \geq 1$. Then (11.18b) gives

$$z_\tau(\lambda) = \sum_{\sigma\trianglerighteq\tau} y_\sigma(\lambda)\mathfrak{l}^\sigma_\tau(w) = \lambda^{2^{-\mu_\tau}}\cdot\eta_\tau + \varkappa_\tau(\lambda),$$

where $\eta_\tau \in W$ and $\|\varkappa_\tau(\lambda)\| = \mathcal{O}\left(\lambda^{2^{-(\mu_\tau-1)}}\right)$. In particular, $z_\tau \sim \lambda^{2^{-\iota_\tau}}$ with

$$\iota_\tau \quad \begin{cases} = \mu_\tau & \text{if} \quad \eta_\tau \neq 0,\\ \in \{0,\dots,\mu_\tau - 1\} & \text{if} \quad \eta_\tau = 0.\end{cases} \tag{11.19}$$

The second case can be avoided, if additionally τ is a critical cell with respect to Γ^1_ϕ. By definition, this implies $\mu_\sigma < \mu_\tau$ for all $\sigma \triangleright \tau$. Hence, (11.19) gives

$$\iota_\sigma \leq \mu_\sigma < \mu_\tau \quad \text{for all} \quad \sigma \triangleright \tau. \tag{11.20}$$

As Ψ and Ξ are mutually inverse on the respective subrepresentations, we obtain

$$\begin{aligned} y_\tau(\lambda)w &= \left(\alpha_\tau\cdot\lambda^{2^{-\mu_\tau}} + \mathcal{O}\left(|\lambda|^{2^{-(\mu_\tau-1)}}\right)\right)\cdot w\\ &= (\Psi((z_\sigma(\lambda))_{\sigma\in\Sigma}))_\tau\\ &= \ell^y_\tau(z_\sigma(\lambda) \mid \sigma\trianglerighteq\tau)\cdot w\\ &= \left(\beta_\tau(\eta_\tau)\cdot\lambda^{2^{-\mu_\tau}} + \mathcal{O}\left(|\lambda|^{2^{-(\mu_\tau-1)}}\right)\right)\cdot w,\end{aligned}$$

where $\beta_\tau\colon W \to \mathbb{R}$ linear. Therein, the last equation holds due to (11.20). Hence, $\beta_\tau(\eta_\tau) = \alpha_\tau \neq 0$ which implies $\eta_\tau \neq 0$. Hence, due to (11.19), $\iota_\tau = \mu_\tau$ so that $z_\tau \sim \lambda^{2^{-\mu_\tau}}$.

Summarizing, we have shown that the representation $(z_\sigma(\lambda))_{\sigma\in\Sigma}$ of the generic branch of steady states $(y_\sigma(\lambda)w)_{\sigma\in\Sigma}$ in $\ker_0(D_\omega\Gamma^{\mathsf{D}}_f(0,0))$ has cell-by-cell asymptotics $z_\sigma \sim \lambda^{2^{-\iota_\sigma}}$. These do not satisfy the ordering (10.26) on all cells. However, for a cell τ that is critical with respect to Γ^1_ϕ, we have that $\iota_\sigma < \iota_\tau$ for all $\sigma \triangleright \tau$. This follows from (11.20) and the fact that $\iota_\tau = \mu_\tau$ for such τ. Due to the bijective conjugation between the dynamics of F and those of the generic original system Γ^{D}_f restricted to its center manifold, each branch $(z_\sigma(\lambda))_{\sigma\in\Sigma}$ of steady states of F uniquely corresponds to a branch of steady states $(w_\sigma(\lambda))_{\sigma\in\Sigma} \subset \mathbf{V}^{\mathsf{D}}$ of Γ_f^{D}. It can be computed as in (11.7)

$$((w_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) = \left(\mathbb{1}_{\underline{\mathcal{X}}^c} + \underline{\psi}\right)\left(Q'\left((z_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda\right)\right) \tag{11.21}$$

using the map whose graph is the center manifold M^c. It remains to be investigated, how the square root orders translate to this representation. From Corollary 11.7 and Remark 11.8 we immediately obtain

$$w_\tau \sim \lambda^{2^{-\iota_\tau}} = \lambda^{2^{-\mu_\tau}} \tag{11.22}$$

for all cells $\tau \in \Sigma$ that are critical with respect to Γ_ϕ^1, as the ordering (10.26) is satisfied for these cells as well as $\iota_\tau = \mu_\tau$.

We cannot apply Corollary 11.7 to cells that are not critical with respect to Γ_ϕ^1. However, using Lemma 11.6, we may take an intermediate step in (11.21) as

$$((X_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) = Q'\left((z_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda\right) \subset \underline{\mathcal{X}}^c \subset \mathbf{V}^{\mathsf{D}} \times \mathbb{R},$$

where $\underline{\mathcal{X}}^c$ is the generalized kernel of the linearization of the extended system at the bifurcation point $D\underline{\Gamma}_f^{\mathsf{D}}(0,0)$ and $Q'\colon \ker_0(D_\omega\Gamma_f^{\mathsf{D}}(0,0)) \times \mathbb{R} \to \underline{\mathcal{X}}^c$ is the isomorphism of Σ-representations given in (11.6). Then $((X_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda)$ is the unique representation of $((z_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda)$ in $\underline{\mathcal{X}}^c$. Due to Lemma 11.6, the X-component of this representation exhibits the same cell-by-cell asymptotics as $(z_\sigma(\lambda))_{\sigma\in\Sigma}$, i.e.

$$X_\sigma \sim \lambda^{2^{-\iota_\sigma}}$$

for all $\sigma \in \Sigma$.

Finally, as in (11.21) we have

$$((w_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda) = \left(\mathbb{1}_{\underline{\mathcal{X}}^c} + \underline{\psi}\right)\left((X_\sigma(\lambda))_{\sigma\in\Sigma}, \lambda\right).$$

Hence, it remains to investigate what impact the map $\left(\mathbb{1}_{\underline{\mathcal{X}}^c} + \underline{\psi}\right)$ has on the square root orders of each cell. Lemma 11.4 does not apply to this situation, as the square root orders for the branch $(X_\sigma(\lambda))_{\sigma\in\Sigma}$ are given by ι_σ for all $\sigma \in \Sigma$ which do not respect the partial order $\trianglelefteq$ as in (10.26). Nevertheless, a proof similar to that of Lemma 11.4 allows for a characterization of the square root orders of the branch $(w_\sigma(\lambda))_{\sigma\in\Sigma}$. Recall from (11.10) that $\underline{\psi}$ is trivial in the λ-component, i.e. $\underline{\psi}(X,\lambda) = (\underline{\psi}_X(X,\lambda), 0)$, and $\underline{\psi}_X$ is a parameter-dependent admissible map of the fundamental network. That is

$$\left(\underline{\psi}_X(X,\lambda)\right)_\tau = q_\tau\left((X_\sigma \mid \sigma \trianglerighteq \tau), \lambda\right).$$

In particular, we compute

$$w_\tau(\lambda) = X_\tau(\lambda) + q_\tau((X_\sigma(\lambda) \mid \sigma \trianglerighteq \tau), \lambda), \tag{11.23}$$

for all $\tau \in \Sigma$, where $q_\sigma(0,0) = 0$ and $dq_\sigma(0,0) = 0$.

Fix a cell $\tau \in \Sigma$ with $\iota_\tau = 0$. Then $\iota_\sigma = 0$ for all $\sigma \triangleright \tau$. Hence, we have $\|X_\sigma(\lambda)\| = \mathcal{O}(|\lambda|)$ for all $\sigma \trianglerighteq \tau$ (which includes τ). Therefore, from (11.23) we obtain

$$\|w_\tau(\lambda)\| = \|X_\tau(\lambda) + q_\tau((X_\sigma(\lambda) \mid \sigma \trianglerighteq \tau), \lambda)\| = \mathcal{O}(|\lambda|),$$

which is equivalent to $w_\tau \sim \lambda$. That is τ has square root order $0 = \iota_\tau = \mu_\tau$.

On the other hand, consider a cell $\tau \in \Sigma$ with $\iota_\tau \geq 1$. We may focus on the case that τ is non-critical with respect to Γ_ϕ^1, as we have investigated the other case already in (11.22). By definition

$$X_\sigma(\lambda) = \lambda^{2^{-\iota_\sigma}} \cdot \vartheta_\sigma + \rho_\sigma(\lambda),$$

for all $\sigma \in \Sigma$, where $\vartheta_\sigma \in W \setminus \{0\}$ and $\|\rho_\sigma(\lambda)\| = \mathcal{O}\left(|\lambda|^{2^{-(\iota_\sigma - 1)}}\right)$. Hence, using (11.23) we obtain

$$w_\tau(\lambda) = X_\tau(\lambda) + q_\tau((X_\sigma(\lambda) \mid \sigma \trianglerighteq \tau), \lambda) = \lambda^{2^{-\iota_\sigma}} \cdot \vartheta_\sigma + \lambda^{2^{-\Upsilon_\tau}} \cdot \varpi_\tau + \Omega_\tau(\lambda),$$

where $\varpi_\tau \in W$, $\|\Omega_\tau(\lambda)\| = \mathcal{O}\left(|\lambda|^{2^{-(\iota_\tau - 1)}} + |\lambda|^{2^{-(\Upsilon_\tau - 1)}}\right)$ and $\Upsilon_\tau = \max_{\sigma \trianglerighteq \tau} \iota_\sigma - 1$. Recall that $\iota_\sigma \leq \mu_\sigma$ for all $\sigma \in \Sigma$. Furthermore,

$$\mu_\tau = \max_{\sigma \triangleright \tau} \mu_\sigma,$$

as τ is non-critical for Γ^1_ϕ. Hence, we obtain

$$\Upsilon_\tau = \max_{\sigma \trianglerighteq \tau} \iota_\sigma - 1 \leq \max_{\sigma \trianglerighteq \tau} \mu_\sigma - 1 = \mu_\tau - 1 < \mu_\tau.$$

This proves $w_\tau \sim \lambda^{2^{-\xi_\tau}}$ where $\xi_\tau \leq \iota_\tau \leq \mu_\tau$. □

Remark 11.13. In Theorems 11.9 and 11.12 we show that the 1-parameter family of admissible vector fields Γ^{D}_f in the case **DD** admits branching steady states that exhibit the same amplification effect as those of Γ^1_ϕ in the case **1D**. In particular, cells that are critical with respect to Γ^1_ϕ exhibit the same asymptotics in both cases – the same holds for cells $\tau \in \Sigma$ with $\mu_\tau = 0$.

On the other hand, consider a cell τ with $\mu_\tau \geq 1$ that is non-critical with respect to Γ^{D}_f. Assume, furthermore, that for a specific branch of steady states $(w_\sigma(\lambda))_{\sigma \in \Sigma}$, we know $w_\sigma \sim \lambda^{2^{-\iota_\sigma}}$ with $\iota_\sigma \geq 0$ for all $\sigma \triangleright \tau$. In order to determine the τ-coordinate of this branch, the equation to be solved is

$$0 = f(w_{\sigma_1 \tau}, \dots, w_{\sigma_n \tau}, \lambda).$$

Note that for all $\sigma \in \Sigma$ either $\sigma\tau = \tau$ or $\sigma\tau \triangleright \tau$. In particular, we may fill in the coordinates of the branch $w_\sigma(\lambda)$ for all $\sigma \triangleright \tau$. The τ equation reduces to a bifurcation equation on W

$$0 = g(w_\tau, \lambda).$$

Due to the bifurcation assumption (B) in Chapter 10, we have

$$D_{w_\tau} g(0,0) = D_{w_\tau} f(0,0) = \sum_{\sigma \in \mathcal{L}_\tau} a_\sigma$$

which is invertible as τ is assumed not to be critical. Therefore, we may apply the implicit function theorem to obtain

$$w_\tau(\lambda) = G((w_\sigma(\lambda) \mid \sigma \triangleright \tau), \lambda)$$

such that

$$0 = g(w_\tau(\lambda), \lambda) = f(w_{\sigma_1 \tau}(\lambda), \dots, w_{\sigma_n \tau}(\lambda), \lambda)$$

for all λ close to 0. As G is at least linear up to lowest order in all its arguments, we immediately obtain

$$w_\tau \sim \lambda^{2^{-\iota_\tau}}$$

with $\iota_\tau \leq \max_{\sigma \triangleright \tau} \iota_\sigma$. In particular, no amplification may occur in a cell that is non-critical with respect to Γ^{D}_f. As a result, all cells that provide an amplification must be critical. Since these are precisely the ones that are critical with respect to Γ^1_ϕ, we obtain that the set of critical cells is the same in both cases. △

Remark 11.14. The bifurcation results in the case **1D** in Chapter 10 are proven by bare-hands analysis exploiting the partial order in the network. The fact that the equation for cell σ depends only on the state variables of cells $\tau \trianglerighteq \sigma$, allows for inductive computations of solutions. A similar approach is possible in the case **DD** as well. Computations in non-critical cells can be solved using the implicit function theorem (compare to Remark 11.13), critical cells require application of the Lyapunov-Schmidt reduction. However, taking care of the numerous different cases of branching patterns, depending on a large number of parameters in the equations, and in particular investigating genericity is at least tedious if not factually impossible, due to the arbitrary dimension of the internal phase space. △

Chapter 12

Example: A feedforward network with 5 cells

We close our investigation of feedforward networks with an example for which we classify the generic 1-parameter steady state bifurcations as predicted by Theorems 10.8, 10.20 and 10.21. In particular, this illustrates the algorithmic computation method in Remark 10.23 and highlights the peculiarities mentioned in Remarks 10.24 and 10.25. We perform the computations for the case of one-dimensional internal dynamics. The results of Chapter 11 imply that the same qualitative branching pattern is generic for arbitrary finite dimensions. In particular the same amplification effect – with the same amplifying cells – can be observed independent of the dimension of the internal phase space. The majority of this chapter is from Section 4.3 in [84].

Consider the feedforward network given by the graph in Figure 12.1. Each arrow color corresponds

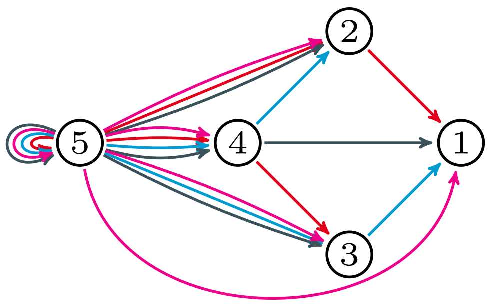


Figure 12.1: A 5-cell feedforward network.

to one input map $\sigma\colon C \to C$. Note that we have not drawn an arrow for $\sigma = \mathrm{Id}$ corresponding to the internal dynamics which we implicitly assume to be there. This network is clearly a feedforward network as it does not contain any loops besides self-loops. Its only maximal cell is cell 5. Furthermore, it possesses two different loop-types $\mathcal{L}_5 = \Sigma$ and $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_3 = \mathcal{L}_4 = \{\mathrm{Id}\}$. Assuming a one-dimensional internal phase space $x_i \in V = \mathbb{R}$ and additional dependence on a real parameter $\lambda \in \mathbb{R}$, the corresponding dynamics is governed by

$$\dot{x} = \gamma_f(x) = \begin{pmatrix} f(x_1, x_2, x_3, x_4, x_5, \lambda) \\ f(x_2, x_5, x_4, x_5, x_5, \lambda) \\ f(x_3, x_4, x_5, x_5, x_5, \lambda) \\ f(x_4, x_5, x_5, x_5, x_5, \lambda) \\ f(x_5, x_5, x_5, x_5, x_5, \lambda) \end{pmatrix}.$$

We want to investigate bifurcations of steady states as in the bifurcation scenario (B) described in

Chapter 10. That is, we assume

$$\gamma_f(0,0) = 0.$$

The linearization at this steady state is

$$D_x\gamma_f(0,0) = \begin{pmatrix} a_{\mathrm{Id}} & a_{\sigma_2} & a_{\sigma_3} & a_{\sigma_4} & a_{\sigma_5} \\ 0 & a_{\mathrm{Id}} & 0 & a_{\sigma_3} & a_{\sigma_2} + a_{\sigma_4} + a_{\sigma_5} \\ 0 & 0 & a_{\mathrm{Id}} & a_{\sigma_2} & a_{\sigma_3} + a_{\sigma_4} + a_{\sigma_5} \\ 0 & 0 & 0 & a_{\mathrm{Id}} & a_{\sigma_2} + a_{\sigma_3} + a_{\sigma_4} + a_{\sigma_5} \\ 0 & 0 & 0 & 0 & a_{\mathrm{Id}} + a_{\sigma_2} + a_{\sigma_3} + a_{\sigma_4} + a_{\sigma_5} \end{pmatrix}.$$

Therein we define $a_\sigma = \partial_\sigma f(0,0)$ (see (10.2)). Furthermore, we follow our convention $\sigma_1 = \mathrm{Id}$. The other partial derivatives are abbreviated accordingly again:

$$\begin{aligned} f_{\sigma\tau} &= \frac{1}{2}\partial_{\sigma\tau} f(0,0), & f_{\sigma\lambda} &= \partial_{\sigma\lambda} f(0,0), \\ \ell &= \partial_\lambda f(0,0), & f_{\lambda\lambda} &= \frac{1}{2}\partial_{\lambda\lambda} f(0,0). \end{aligned}$$

According to Theorem 9.1 the eigenvalues of the linearization are in one-to-one correspondence with the loop types of the network. This can also easily be read off of the matrix. As a matter of fact the linearization has two eigenvalues $\sum_{\sigma\in\mathcal{L}_5} a_\sigma = a_{\mathrm{Id}} + a_{\sigma_2} + a_{\sigma_3} + a_{\sigma_4} + a_{\sigma_5}$, which is simple, and $\sum_{\sigma\in\mathcal{L}_1} a_\sigma = a_{\mathrm{Id}}$ which has algebraic multiplicity 4. For a steady state bifurcation to occur, the linearization has to have an eigenvalue 0. Generically – i.e. for a generic choice of system parameters –, in such a point only one of the two eigenvalues vanishes. Under these assumptions we investigate generic solutions to

$$\gamma_f(x,\lambda) = 0$$

close to the bifurcation point.

Let us investigate the case $\sum_{\sigma\in\mathcal{L}_5} a_\sigma = a_{\mathrm{Id}} + a_{\sigma_2} + a_{\sigma_3} + a_{\sigma_4} + a_{\sigma_5} = 0$ and $\sum_{\sigma\in\mathcal{L}_1} a_\sigma = a_{\mathrm{Id}} \neq 0$ first. In particular, this means the maximal cell is critical while all other cells are not. As a result, all branches of steady state solutions are given in Theorem 10.8. As there is only one maximal cell, necessarily all branches are fully synchronous. We obtain two different saddle node branches depending on the system parameters. If

$$\frac{\ell}{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}} < 0$$

we compute

$$x_i = \pm\sqrt{-\frac{\ell}{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}}} \cdot \sqrt{\lambda} + \mathcal{O}(|\lambda|).$$

for $i = 1, \dots, 5$. Note that therein the choice of sign is the same for all cells simultaneously yielding exactly two fully synchronous branches. On the other hand, if

$$\frac{\ell}{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}} > 0$$

we obtain

$$x_i = \pm\sqrt{\frac{\ell}{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}}} \cdot \sqrt{-\lambda} + \mathcal{O}(|\lambda|).$$

for $i = 1, \dots, 5$ accordingly. These branches exists for $|\lambda|$ small. Generically, no other cases are possible so that no other branching solutions exist.

Next, we turn to the case $K = \sum_{\sigma\in\mathcal{L}_5} a_\sigma = a_{\mathrm{Id}} + a_{\sigma_2} + a_{\sigma_3} + a_{\sigma_4} + a_{\sigma_5} \neq 0$ and $\sum_{\sigma\in\mathcal{L}_1} a_\sigma = a_{\mathrm{Id}} = 0$. Equivalently the maximal cells are not critical but all the other cells are. Theorems 10.20 and 10.21

provide all generic branching solutions. As a first step, we have to determine all possible blocks $B \subset C$ such that $p \notin B$ but $q \in B$ for all $q \rhd p$ implies p critical (see Remark 10.23). The possible choices are $\{5\}, \{4,5\}, \{3,4,5\}, \{2,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}$. For each solution branch the cells in exactly one of these blocks are in the fully synchronous state $X(\lambda)$ as in Lemma 10.17 while all others are not. The solutions for the variables of the remaining cells are computed iteratively with respect to the partial order $\trianglelefteq$ according to the rules in Theorems 10.20 and 10.21. If for a cell p the existence condition in these theorems is not satisfied for all choices of coefficients for cells $q \rhd p$, this implies that the entire solution branch does not exist. We describe the branches as briefly as possible starting with the most simple case.

Assume $B = \{1,2,3,4,5\}$. All cells remain in the fully synchronous state, i.e.

$$x_i(\lambda) = X(\lambda) = -\frac{\ell}{K}\lambda + \mathcal{O}\left(|\lambda|^2\right)$$

for $i = 1, \dots, 5$. This branch exists for $|\lambda|$ small independent of the sign and without any further restrictions on the system parameters.

Next, assume $B = \{2,3,4,5\}$. We obtain

$$x_5(\lambda) = x_4(\lambda) = x_3(\lambda) = x_2(\lambda) = X(\lambda) = -\frac{\ell}{K}\lambda + \mathcal{O}\left(|\lambda|^2\right).$$

As cell 1 is critical but not in the fully synchronous state, this leaves

$$x_1(\lambda) = \left(\frac{\ell}{K}\left(1 + 2\frac{f_{\mathrm{Id}\,\sigma_2} + f_{\mathrm{Id}\,\sigma_3} + f_{\mathrm{Id}\,\sigma_4} + f_{\mathrm{Id}\,\sigma_5}}{f_{\mathrm{Id}\,\mathrm{Id}}}\right) - \frac{f_{\mathrm{Id}\,\lambda}}{f_{\mathrm{Id}\,\mathrm{Id}}}\right)\lambda + \mathcal{O}\left(|\lambda|^2\right).$$

This branch exists without any further restrictions on the system parameters as well.

Consider $B = \{3,4,5\}$ and focus on $\lambda > 0$. We obtain

$$x_5(\lambda) = x_4(\lambda) = x_3(\lambda) = X(\lambda) = -\frac{\ell}{K}\lambda + \mathcal{O}\left(|\lambda|^2\right)$$

and abbreviate $D_5 = D_4 = D_3 = -\ell/K$. As cell 2 is critical but $2 \notin B$, we obtain

$$\begin{aligned} x_2(\lambda) &= \left(\frac{\ell}{K}\left(1 + 2\frac{f_{\mathrm{Id}\,\sigma_2} + f_{\mathrm{Id}\,\sigma_3} + f_{\mathrm{Id}\,\sigma_4} + f_{\mathrm{Id}\,\sigma_5}}{f_{\mathrm{Id}\,\mathrm{Id}}}\right) - \frac{f_{\mathrm{Id}\,\lambda}}{f_{\mathrm{Id}\,\mathrm{Id}}}\right)\lambda + \mathcal{O}\left(|\lambda|^2\right) \\ &= D_2\lambda + \mathcal{O}\left(|\lambda|^2\right). \end{aligned}$$

Then cell 1 is critical and receives an input from a cell not in B. Thus, we have to distinguish two cases. If

$$(*) = \left(a_{\sigma_2}D_2 + a_{\sigma_3}D_3 + a_{\sigma_4}D_4 + a_{\sigma_5}D_5 + \ell\right) f_{\mathrm{Id}\,\mathrm{Id}} < 0,$$

we obtain

$$x_1(\lambda) = \pm\sqrt{-\frac{a_{\sigma_2}D_2 + a_{\sigma_3}D_3 + a_{\sigma_4}D_4 + a_{\sigma_5}D_5 + \ell}{f_{\mathrm{Id}\,\mathrm{Id}}}} \cdot \sqrt{\lambda} + \mathcal{O}(|\lambda|).$$

If, on the other hand, $(*) > 0$, there is no solution branch with $B = \{3,4,5\}$. However, for this choice of B we obtain the same solutions for cells $2,3,4,5$ for $\lambda < 0$. They can be written as $x_i(\lambda) = E_i \cdot (-\lambda) + \mathcal{O}(|\lambda|^2)$, where $E_i = -D_i$ (compare to Remark 10.25). The condition for the existence of a branching solution for cell 1 is

$$\left(a_{\sigma_2}E_2 + a_{\sigma_3}E_3 + a_{\sigma_4}E_4 + a_{\sigma_5}E_5 - \ell\right) f_{\mathrm{Id}\,\mathrm{Id}} < 0.$$

Note that the left hand side of this inequality is $-(*)$. Hence, if $(*) > 0$, we obtain

$$x_1(\lambda) = \pm\sqrt{\frac{a_{\sigma_2}D_2 + a_{\sigma_3}D_3 + a_{\sigma_4}D_4 + a_{\sigma_5}D_5 + \ell}{f_{\mathrm{Id}\,\mathrm{Id}}}} \cdot \sqrt{-\lambda} + \mathcal{O}(|\lambda|).$$

If $(*) < 0$ there is no solution for cell 1 for $\lambda < 0$. Summarizing we see that depending on the sign of $(*)$, the branching solutions for cell 1 exist for precisely one sign of λ. Hence, a branch of steady states for $B = \{3, 4, 5\}$ exists either only for $\lambda > 0$ or only for $\lambda < 0$.

The considerations for $B = \{2, 4, 5\}$ are almost identical to those made for $B = \{3, 4, 5\}$. Exchanging cells 2 and 3 as well as the input maps σ_2 and σ_3 provides the solution branches.

The case $B = \{4, 5\}$ is very similar as well. Cells 4 and 5 remain in the fully synchronous state $X(\lambda)$. More precisely for cells $i = 2, \ldots, 5$ we obtain

$$x_i(\lambda) = D_i\lambda + \mathcal{O}\left(|\lambda|^2\right) = E_i \cdot (-\lambda) + \mathcal{O}\left(|\lambda|^2\right)$$

with

$$\begin{aligned}
D_5 &= -E_5 = D_4 = -E_4 = -\frac{\ell}{K}, \\
D_3 &= -E_3 = D_2 = -E_2 = \frac{\ell}{K}\left(1 + 2\frac{f_{\mathrm{Id}\,\sigma_2} + f_{\mathrm{Id}\,\sigma_3} + f_{\mathrm{Id}\,\sigma_4} + f_{\mathrm{Id}\,\sigma_5}}{f_{\mathrm{Id}\,\mathrm{Id}}}\right) - \frac{f_{\mathrm{Id}\,\lambda}}{f_{\mathrm{Id}\,\mathrm{Id}}}
\end{aligned}$$

for $\lambda > 0$ and $\lambda < 0$ respectively. Similar to before we obtain

$$\begin{aligned}
x_1(\lambda) &= \pm\sqrt{-\frac{a_{\sigma_2}D_2 + a_{\sigma_3}D_3 + a_{\sigma_4}D_4 + a_{\sigma_5}D_5 + \ell}{f_{\mathrm{Id}\,\mathrm{Id}}}} \cdot \sqrt{\lambda} + \mathcal{O}(|\lambda|) \quad \text{or} \\
x_1(\lambda) &= \pm\sqrt{\frac{a_{\sigma_2}D_2 + a_{\sigma_3}D_3 + a_{\sigma_4}D_4 + a_{\sigma_5}D_5 + \ell}{f_{\mathrm{Id}\,\mathrm{Id}}}} \cdot \sqrt{-\lambda} + \mathcal{O}(|\lambda|)
\end{aligned}$$

for $\lambda > 0$ or $\lambda < 0$ respectively, if

$$\left(a_{\sigma_2}D_2 + a_{\sigma_3}D_3 + a_{\sigma_4}D_4 + a_{\sigma_5}D_5 + \ell\right) f_{\mathrm{Id}\,\mathrm{Id}} < 0 \quad \text{or} \quad > 0.$$

Once again, a solution branch for this choice of B exists for precisely one sign of λ.

Finally, we investigate the case $B = \{5\}$. The mechanism that relates the two cases – i.e. branching solutions for $\lambda > 0$ and $\lambda < 0$ – is the same as in the previous cases. Therefore we omit the computational details. Cell 5 remains in the fully synchronous state

$$x_5(\lambda) = X(\lambda) = D_5\lambda + \mathcal{O}\left(|\lambda|^2\right) = -\frac{\ell}{K}\lambda + \mathcal{O}\left(|\lambda|^2\right).$$

For cell 4 we obtain

$$\begin{aligned}
x_4(\lambda) &= D_4\lambda + \mathcal{O}\left(|\lambda|^2\right) \\
&= \left(\frac{\ell}{K}\left(1 + 2\frac{f_{\mathrm{Id}\,\sigma_2} + f_{\mathrm{Id}\,\sigma_3} + f_{\mathrm{Id}\,\sigma_4} + f_{\mathrm{Id}\,\sigma_5}}{f_{\mathrm{Id}\,\mathrm{Id}}}\right) - \frac{f_{\mathrm{Id}\,\lambda}}{f_{\mathrm{Id}\,\mathrm{Id}}}\right)\lambda + \mathcal{O}\left(|\lambda|^2\right).
\end{aligned}$$

Considering cell 3, we obtain

$$\begin{aligned}
x_3(\lambda) &= \pm\sqrt{-\frac{a_{\sigma_2}D_4 + (a_{\sigma_3} + a_{\sigma_4} + a_{\sigma_5})D_5 + \ell}{f_{\mathrm{Id}\,\mathrm{Id}}}} \cdot \sqrt{\lambda} + \mathcal{O}(|\lambda|) \quad \text{or} \\
x_3(\lambda) &= \pm\sqrt{\frac{a_{\sigma_2}D_4 + (a_{\sigma_3} + a_{\sigma_4} + a_{\sigma_5})D_5 + \ell}{f_{\mathrm{Id}\,\mathrm{Id}}}} \cdot \sqrt{-\lambda} + \mathcal{O}(|\lambda|),
\end{aligned}$$

if

$$(*) = \left(a_{\sigma_2}D_4 + (a_{\sigma_3} + a_{\sigma_4} + a_{\sigma_5})D_5 + \ell\right) f_{\mathrm{Id}\,\mathrm{Id}} < 0 \quad \text{or} \quad > 0$$

respectively. In particular, $B = \{5\}$ does not provide a solution branch for $\lambda > 0$, if $(*) > 0$, or for $\lambda < 0$, if $(*) < 0$. Similarly, we obtain

$$x_2(\lambda) = \pm\sqrt{-\frac{a_{\sigma_3}D_4 + (a_{\sigma_2} + a_{\sigma_4} + a_{\sigma_5})D_5 + \ell}{f_{\mathrm{Id\,Id}}}} \cdot \sqrt{\lambda} + \mathcal{O}(|\lambda|) \quad \text{or}$$
$$x_2(\lambda) = \pm\sqrt{\frac{a_{\sigma_3}D_4 + (a_{\sigma_2} + a_{\sigma_4} + a_{\sigma_5})D_5 + \ell}{f_{\mathrm{Id\,Id}}}} \cdot \sqrt{-\lambda} + \mathcal{O}(|\lambda|),$$

if

$$(**) = (a_{\sigma_3}D_4 + (a_{\sigma_2} + a_{\sigma_4} + a_{\sigma_5})D_5 + \ell)\, f_{\mathrm{Id\,Id}} < 0 \quad \text{or} \quad > 0$$

respectively. In particular, $B = \{5\}$ does not provide a solution branch for $\lambda > 0$, if $(**) > 0$, or for $\lambda < 0$, if $(**) < 0$. Hence, if $(*)$ and $(**)$ have opposite signs, neither of the two branches exists. If both have the same sign, we abbreviate the lowest order coefficients as $\pm D_3, \pm D_2, \pm E_3, \pm E_2$ as before. We only need to investigate cell 1 in that case. Consider $(*), (**) < 0$. If

$$(***) = (\pm a_{\sigma_2}D_2 \pm a_{\sigma_3}D_3)\, f_{\mathrm{Id\,Id}} < 0,$$

we obtain

$$x_1(\lambda) = \pm\sqrt{-\frac{\pm a_{\sigma_2}D_2 \pm a_{\sigma_3}D_3}{f_{\mathrm{Id\,Id}}}}\sqrt{\sqrt{\lambda}} + \mathcal{O}\left(\sqrt{|\lambda|}\right).$$

If $(***) > 0$, the solution branch does not exist. Note that $(***)$ depends on the choice of signs for the coefficients in cells 2 and 3. Therefore not all of them might be admissible to provide a solution branch. More precisely, exactly half of the possible choices yields a negative sign of $(***)$ while the other half yields a positive sign. This is due to the fact that $(***)$ and $-(***)$ are both possible choices, while $(***) \neq 0$ generically. Similarly, for $(*), (**) > 0$ we obtain

$$x_1(\lambda) = \pm\sqrt{-\frac{\pm a_{\sigma_2}D_2 \pm a_{\sigma_3}D_3}{f_{\mathrm{Id\,Id}}}}\sqrt{\sqrt{-\lambda}} + \mathcal{O}\left(\sqrt{|\lambda|}\right),$$

if

$$(\pm a_{\sigma_2}D_2 \pm a_{\sigma_3}D_3)\, f_{\mathrm{Id\,Id}} < 0$$

for admissible choices of signs.

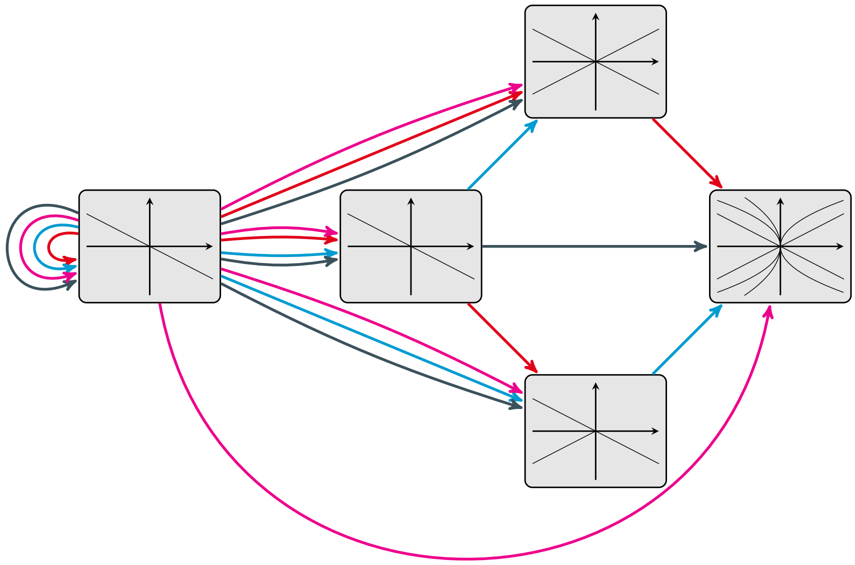
We have therefore computed all branches of steady states in a generic 1-parameter bifurcation in the case of one-dimensional internal dynamics. Note that the network in Figure 12.1 is its own fundamental network. Hence, from Theorems 11.9 and 11.12 we see that the generic steady state bifurcation branches for high-dimensional internal dynamics are qualitatively essentially the same. In particular, for a branch of steady states in a generic steady state bifurcation with high-dimensional internal dynamics we obtain that each cell has the same square root order as in one of the branches we have computed for the one-dimensional case before. Only for the non-critical cells, the square root order might be lower.

Finally we make some remarks about the peculiarities concerning existence and non-existence of certain branches of steady states. We see that there are numerous ways in which a solution branch with $B = \{5\}$ fails to exist. These ultimately depend on the system parameters. Hence, there are different solutions in different regions of system parameter space. We briefly introduce two cases to illustrate that already this simple network produces unexpected – compared to the informal introduction of the amplification effect – bifurcation scenarios.

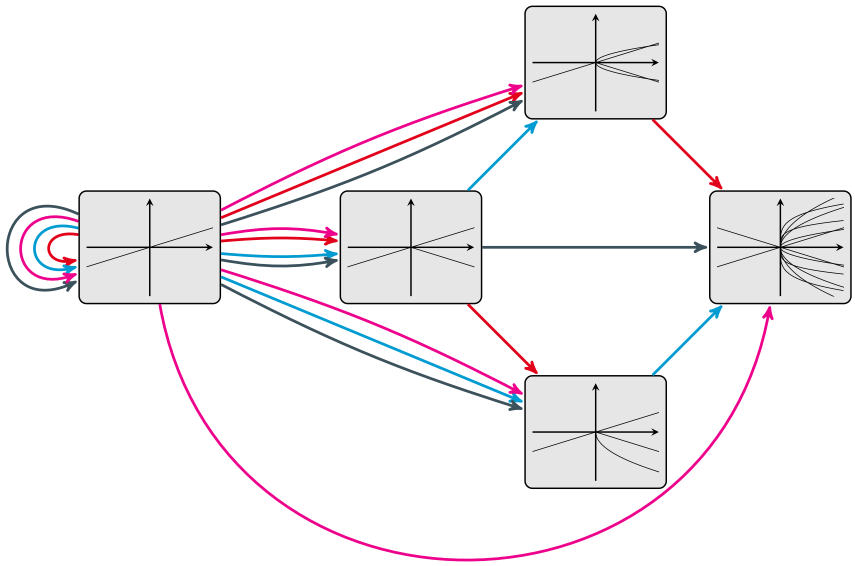
Consider the bifurcation scenario as before with $a_{\sigma_3} = -2a_{\sigma_2}$ as well as $a_{\sigma_3} = a_{\sigma_4} = 0$ and investigate $B = \{5\}$. We compute $(**) = -2 \cdot (*)$ proving that generically $(*)$ and $(**)$ have opposite signs. Therefore, there are no branching solutions with $B = \{5\}$, as cell 2 forces the branch to exist for $\lambda > 0$ and cell 3 forces it to exist for $\lambda < 0$ or the other way around. This implies the existence of an open region in parameter space for which this issue occurs. The reason lies in the structure of the network. The two cells 2 and 3 receive the same inputs. However, the input from cell 4 comes via different arrow types. As these types reflect various types of interactions, this can lead to one cell only amplifying its inputs 'before' the bifurcation point and the other one 'after' the bifurcation point $\lambda = 0$.

On the other hand, whenever $(*)$ and $(**)$ have the same sign, there is also a suitable choice of coefficients in cells 2 and 3 such that $(***) < 0$, as was mentioned before. Hence, there is also generically a branching solution for cell 1 and the resulting in the generic existence of the entire solution branch with $B = \{5\}$.

In Figure 12.2 we illustrate the steady state bifurcations for two different choices of parameter values. The qualitative bifurcation scenario is depicted for each cell separately. Note that for a non-maximal cell certain branches are only possible if cells above it are in a suitable state. This fact is not displayed in the figures. Both choices of parameters are generic but display different behavior. The amplification effect can be seen in both. However, in Figure 12.2a the strongest amplification is $\sim \sqrt{\lambda}$ in cell 1, whereas we also find a branch $\sim \sqrt[4]{\lambda}$ in cell 1 in Figure 12.2b.



(a) Parameters: $a_{\sigma_1} = 0, a_{\sigma_2} = 1, a_{\sigma_3} = -2, a_{\sigma_4} = 1, a_{\sigma_5} = 1, \ell = 1, f_{\mathrm{Id}\,\mathrm{Id}} = -1, f_{\mathrm{Id}\,\lambda} = 1, f_{\mathrm{Id}\,\sigma_2} + f_{\mathrm{Id}\,\sigma_3} + f_{\mathrm{Id}\,\sigma_4} + f_{\mathrm{Id}\,\sigma_5} = \frac{1}{2}$.



(b) Parameters: $a_{\sigma_1} = 0, a_{\sigma_2} = 2, a_{\sigma_3} = 1, a_{\sigma_4} = -1, a_{\sigma_5} = 0, \ell = -1, f_{\mathrm{Id}\,\mathrm{Id}} = 1, f_{\mathrm{Id}\,\lambda} = 0, f_{\mathrm{Id}\,\sigma_2} = 0, f_{\mathrm{Id}\,\sigma_3} = 0, f_{\mathrm{Id}\,\sigma_4} = 0, f_{\mathrm{Id}\,\sigma_5} = 0$.

Figure 12.2: Depiction of the qualitative steady state bifurcations of the network in Figure 12.1 with different parameter values. The diagrams describe each cells behavior separately. However, the branching is not independent of the other cells as described in Chapter 12.

Part IV

Conclusions

Contents

Chapter 13

The algebra in network dynamics

In this thesis we have seen how the usage of algebraic techniques can help to address the issues – or at least certain aspects thereof – that emerge in the investigation of network dynamical systems (see Chapter 1) and to characterize dynamical phenomena that are inherent to a given network. This is of course not surprising as networks are inherently algebraic objects. Most notably, they are modular in nature – i.e. naturally made up of smaller building blocks. In particular, groupoids allow to characterize robust patterns of synchrony in terms of balanced equivalence relations, the classical reduction methods leave the structure of semigroup networks intact, graph fibrations relate the dynamical properties of different networks and, as a result, introduce hidden symmetries, and operad and sheaf theory flexibly encode the modular nature of networks.

We focus our work on the formalism of semigroup networks. These prove to be particularly powerful due to their well-behavedness. An abundance of algebraic properties and objects can be related to networks of this type which provide tools to encode certain aspects of the network structure in ways that are respected by classical methods of dynamical analysis. Most importantly, the interaction structure is entirely encoded in a semigroup or monoid of input maps. Hence, investigation of structural properties of a network amounts to an algebraic investigation of the structure of the corresponding monoid. For example feedforward structure is equivalent to the monoid being L-trivial which yields a class of monoids that is also studied in the algebraic field of monoid theory. Furthermore, the fundamental network construction, which exclusively relies on the monoid of input maps, gives rise to hidden symmetries that, unlike the classical interpretation, are not actual symmetries of the (fundamental) network but more general transformations. Nevertheless, they introduce equivariance of the admissible vector fields in the same way as classical symmetries do. However, in this case equivariance respects the more general transformations which are encoded in a representation of the monoid of input maps. First and foremost this observation has led to a significant generalization of the theory of equivariant dynamics which focused on finite or compact groups before. Fortunately, most of the well-established machinery has – more or less direct – generalizations to more general types of symmetries. Most notably, we mention the reduction methods in terms of normal forms, the Lyapunov-Schmidt reduction, and the center manifold reduction. Interestingly, the latter can only be generalized to monoid equivariant systems in the context of fundamental networks as it crucially depends on the response function that drives the internal dynamics of each cell. Nonetheless, all of these methods can be applied to a fundamental network and additionally, via graph fibrations, also have their impact on the original network. These relations allow to classify the synchrony breaking bifurcations that are inherent to a given semigroup network – i.e. forced by the network structure but not by a specific choice of governing functions – following a standardized step-by-step machinery (see Section 6.1). We were able to provide the crucial step of classifying spectral properties in synchrony breaking steady state bifurcations in terms of monoid representations in Theorem 5.11.

The implications of the algebraic nature of networks go beyond exploiting hidden symmetries for the classification of synchrony breaking bifurcations. Once again, consider feedforward networks as an example. This structural property is not only equivalent to the monoid of input maps being L-trivial but also the existence of a partial order of the cells of the network in terms of mutual dependence. This is reflected in the system of ordinary differential equations as well, so that using knowledge about finite partially ordered sets allows for an easy (conceptually, but technically complex) classification of synchrony breaking steady state bifurcations by solving suitable equations inductively with respect to the partial order. On the other hand, the partial order can equivalently be observed in the (L-trivial) monoid of input maps which allows for a new classification of the algebraic property L-triviality.

In general, not only are algebraic methods necessary tools in the analysis of network dynamics but observations from network dynamics can also stimulate development in the corresponding areas of algebra. One of the driving mechanisms behind this is the fact that the fundamental network can equivalently be characterized as the regular module of the monoid algebra of the monoid of input maps (see Section 4.2). The monoid algebra is an object that is commonly studied in representation theory of monoids to deduce algebraic properties of a given representation while the regular module allows to investigate an algebra as a module. Most importantly, we currently understand this construction as the algebraic explanation of the hidden symmetries of the fundamental networks as well as the structure of its linear admissible maps – i.e. the endomorphisms of the corresponding representation space. This observation has motivated the approach to characterize possible generalized kernels in bifurcation problems in terms of equivariant projection operators. It is currently an open question whether it can furthermore provide insight into other parts of the analysis of network dynamics – in particular synchrony breaking bifurcations. It was also the regular module of the monoid algebra that inspired the most apparent example of a purely algebraic result induced by network dynamics. Using the fact that the monoid representation of a feedforward fundamental network can be decomposed into subrepresentations exploiting the partial order mentioned above as well as the duality of notions of L- and R-trivial monoids, one can deduce an algorithm to find a complete system of primitive orthogonal idempotents in an R-trivial monoid algebra (see Remark 9.9). This is a purely algebraic result in monoid representation theory. Even if the algorithm does not solve an open problem therein, the fact that it has been addressed before reveals that it is a relevant observation nonetheless. Furthermore, the network inspired approach allows for simpler computation in a more general setting, underlining its worth possibly also for the specialists in the field.

Summarizing, we have seen how the introduction of purely algebraic methods into the study of network dynamical systems – not restricted to semigroup networks – has proven to be a successful strategy to tackle the issues and problems that network structure imposes. Many results have already been proven, phenomena have been explained and recast in algebraic language, and more results in that general direction are to be expected. On the other hand, taking the network point of view into algebra can also provide new insight into known or previously unexplored problems therein. New structures are being investigated or known ones gain importance from applications in network dynamics. We conclude, that the investigation of synergy between networks and algebra is certainly worth further pursuit and provide an outlook into one possible direction in the following final chapter.

Chapter 14

Outlook: Encoding network structure in quiver representations

An objective of investigation in a current joint research project with Bob Rink and Eddie Nijholt is to determine which 'types' of interaction structures can be encoded algebraically in terms of generalized symmetry. Throughout this thesis we have seen – and used at many points – that the fundamental network structure of a semigroup network is equivalent to a specific representation of a monoid – which is a generalization of classical (compact or finite) group symmetries. This construction, however, is only applicable to the class of (homogeneous) networks with asymmetric inputs whose input maps form a monoid. This suggests the question whether even less restrictive algebraic objects can be used to model more general network structures – or at least parts of it such as the information which cells a given one depends on. Promising objects to study in this regard are *quiver representations*. As we have seen in Section 2.4, graph fibrations – that is maps sending one network to another while respecting interaction structure – can be used encode all kinds of structural properties of a network. We have used these maps almost exclusively for the relation between a network and its fundamental network as well as for the symmetries of the latter in terms of self-fibrations. But more general features can be displayed as well. In Example 2.7 we encode feedforward structure in terms of an injective graph fibration of a maximal cell – in the language of Part III – into the 2-cell feedforward chain. The corresponding semiconjugacy then encodes the fact that the first cell does not depend on the second in the admissible vector fields. Note that this can be generalized to general feedforward structure. A quiver representation allows to describe all these relations in terms of a single algebraic object.

Let us make this slightly more precise by restating some definitions from the nice introduction Krause [67] to which we also refer for background on the matter (we thank Eddie Nijholt for making us aware of this text).

Definition 14.1. *A* quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, s, t)$ *is a finite directed graph where* $\mathcal{Q}_0$ *are the* vertices*,* $\mathcal{Q}_1$ *are the* arrows*, and the two maps* $s, t\colon \mathcal{Q}_1 \to \mathcal{Q}_0$ *are such that an arrow* $\alpha \in \mathcal{Q}_1$ starts *at* $s(\alpha)$ *and* terminates *at* $t(\alpha)$*.*

Informally speaking, the objects we are interested in are finite directed graphs without any further restrictions. In particular, there are no 'types' or 'colors' of cells or arrows. Nevertheless, a quiver is in some sense a network itself. However, they are used with an entirely different intention.

Definition 14.2. *A* representation *of the quiver* $\mathcal{Q}$ *is a collection*

$$\mathbf{V} = (\mathbf{V}_i, L_\alpha)_{i\in\mathcal{Q}_0, \alpha\in\mathcal{Q}_1},$$

where $\mathbf{V}_i$ *is a (finite-dimensional) vector space assigned to each vertex* $i \in \mathcal{Q}_0$ *and* $L_\alpha\colon \mathbf{V}_{s(\alpha)} \to \mathbf{V}_{t(\alpha)}$ *is a linear map from the vector space assigned to the starting vertex of* α *to the one assigned to the terminating vertex of* α*.*

Contrary to the total phase spaces of networks – or more precisely the collection of internal phase spaces –, a quiver representation contains information about interactions in terms of the linear maps L_α which relate the vector spaces that are attached to the vertices of the quiver with respect to its arrows. Note that it is also the linear maps L_α for each arrow that make the collection of vector spaces an actual representation. In a more classical representation, say of a monoid, each element acts as a linear map on the representation space of which there is only one. Conversely, in a quiver representation each arrow acts via a linear map. Due to the flexibility induced by different representation spaces for each vertex, these now have to map one of these space to another according to the direction of the arrow.

This is also what makes quiver representations relevant for the investigation of network dynamics. However, perhaps surprisingly, instead of describing one network and its internal phase spaces in these terms we encode multiple networks and their relations as a quiver representation. That is, it turns out to be convenient to define a network of networks as a quiver and the collection of total phase spaces as its representation. Recall from Section 2.4 and Chapter 3 that a graph fibration $\varphi\colon \mathcal{N}_1 \to \mathcal{N}_2$ gives rise to a a linear map $\varphi^*\colon \mathbf{V}_{\mathcal{N}_2} \to \mathbf{V}_{\mathcal{N}_1}$, where we abbreviate the total phase spaces as $\mathbf{V}_{\mathcal{N}_1}$ and $\mathbf{V}_{\mathcal{N}_2}$. Then we may define a quiver $\mathcal{Q}$ as follows: the vertices are given by the networks $\mathcal{N}_1$ and $\mathcal{N}_2$ and there is one arrow α_φ in the opposite direction of φ, i.e. from $\mathcal{N}_2$ to $\mathcal{N}_1$. Reversing the direction of the arrow is necessary in order to be consistent with the definition of a quiver representation. It is easy to see that the collection of total phase spaces $\mathbf{V}_{\mathcal{N}_1}, \mathbf{V}_{\mathcal{N}_2}$ together with the linear map $L_{\alpha_\varphi} = \varphi^*$ satisfies the condition of Definition 14.2. That is, it is a representation of the quiver $\mathcal{Q} = (\{\mathcal{N}_1, \mathcal{N}_2\}, \{\alpha_\varphi\}, s\colon \alpha_\varphi \mapsto \mathcal{N}_2, t\colon \alpha_\varphi \mapsto \mathcal{N}_1)$. This observation is visualized in Figure 14.1. The formalism has the convenient feature that the quiver $\mathcal{Q}$ – and with it its representation – can readily be extended to include more vertices. If there is another graph fibration $\overline{\varphi}\colon \mathcal{N}_2 \to \mathcal{N}_3$, we may extend $\mathcal{Q}$ to also include the arrow $\alpha_{\overline{\varphi}}$. The corresponding semiconjugacy extends the representation accordingly. Hence, if we investigate a network $\mathcal{N}$ we may encode graph fibrations to any network in terms of the quiver representation. But we can also only include specific ones. This is particularly important, as usually not all possible graph fibrations of a given network into another one (or the other way around) are known.

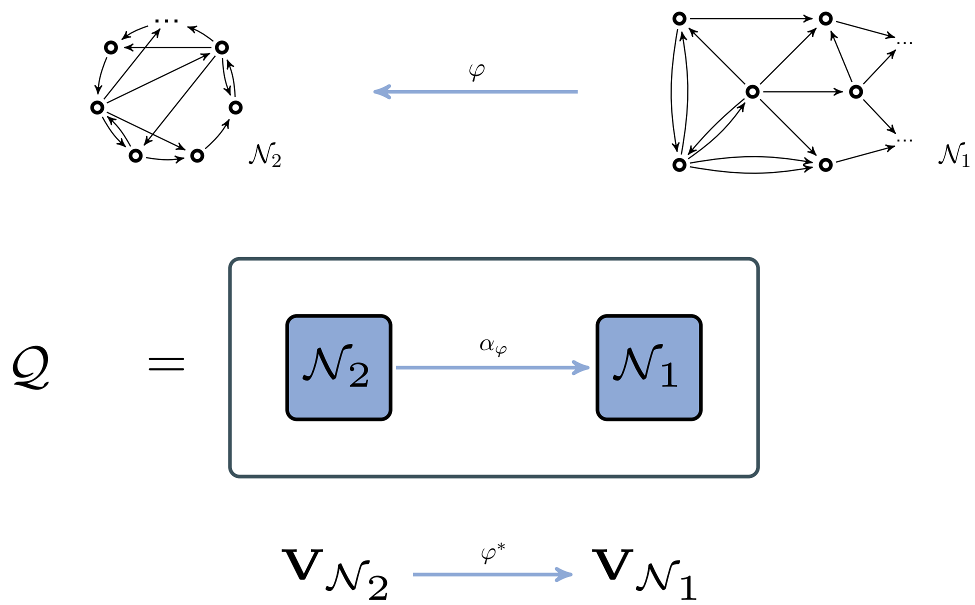


Figure 14.1: Visualization of a quiver representation induced by a graph fibration.

Apart from encoding structural properties of networks, quiver representations also have an impact on dynamical systems on the corresponding representation spaces. In particular, the linear map induced by a graph fibration is not only a map between the total phase spaces but most importantly it induces a linear semiconjugacy of admissible vector fields. In the above setting let $\gamma_f^{\mathcal{N}_1}$ and $\gamma_f^{\mathcal{N}_2}$ be admissible vector fields of $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively that are governed by the same response function – this is possible as the existence of a graph fibration implies that both networks have the same input trees / input networks. Then

$$\varphi^* \circ \gamma_f^{\mathcal{N}_2} = \gamma_f^{\mathcal{N}_1} \circ \varphi^*. \tag{14.1}$$

This motivates the generalized definition of equivariance.

Definition 14.3. *Let $\mathcal{Q}$ be a quiver and let $\mathbf{V} = (\mathbf{V}_i, L_\alpha)_{i\in\mathcal{Q}_0,\alpha\in\mathcal{Q}_1}$ be a representation of $\mathcal{Q}$. A collection of smooth vector fields*

$$F = (F_i)_{i\in\mathcal{Q}_0}$$

with $F_i\colon \mathbf{V}_i \to \mathbf{V}_i$ is called equivariant with respect to the representation*, if*

$$L_\alpha \circ F_{s(\alpha)} = F_{t(\alpha)} \circ L_\alpha \tag{14.2}$$

for all arrows $\alpha \in \mathcal{Q}_1$. That is, a collection of smooth vector fields, one for each vertex, that are semiconjugate by the representation maps.

The linear semiconjugacies induced by graph fibrations satisfy this definition (compare (14.1) and (14.2)). Furthermore, note that we may define homomorphisms, endomorphisms, and isomorphisms of quiver representations accordingly each time consisting of a collection of linear maps of the vector spaces attached to each vertex.

Without explicit mention we have encountered quiver representations and equivariant vector fields in the context of non-homogeneous networks with asymmetric inputs (Section 3.6). For these we compute a collection of fundamental networks – one for each type of cells in the original network – and a collection of graph fibrations in between. Then a collection of vector fields on the total phase spaces is admissible for the fundamental networks if and only if it is equivariant as in Definition 14.3 with respect to the collection of graph fibrations. Particular importance of this observation stems from the equivalence part. The fundamental network structure can equivalently be encoded in terms of equivariance with respect to generalized symmetries. This is also what makes the fundamental network construction for semigroup networks so powerful. In the more general context when a quiver encodes relations between different networks we cannot expect to obtain an equivalent characterization of admissibility in terms of equivariance. Especially since we do not necessarily encode all structural properties in terms of graph fibrations. Nevertheless, equivariance of a collection of vector fields with respect to the quiver representation is equivalent to the fact that these vector fields respect all structural features that are encoded in the quiver. To make this more tangible recall Example 2.7. We are interested in the network $\mathcal{N}_1$ in Figure 14.1 and observe the graph

Figure 14.2: The injective graph fibration that encodes a 2-cell feedforward chain.

fibration $\varphi\colon \mathcal{N}_2 \to \mathcal{N}_1$ that sends cell 1 to cell 1 and its self-loop to the self-loop. The networks are homogeneous. Hence we attach a finite-dimensional internal phase space V to each cell. Then the total phase spaces are $\mathbf{V}_{\mathcal{N}_1} = V^2$ and $\mathbf{V}_{\mathcal{N}_2} = V$ and the graph fibration φ induces the linear map

$$\begin{aligned} \varphi^*\colon V^2 &\to V \\ (x_1, x_2) &\mapsto x_{\varphi(1)} = x_1. \end{aligned}$$

As before, (V^2, V, φ^*) is a representation of the quiver

$$\mathcal{Q} = (\{\mathcal{N}_1, \mathcal{N}_2\}, \{\alpha_\varphi\}, s\colon \alpha_\varphi \mapsto \mathcal{N}_2, t\colon \alpha_\varphi \mapsto \mathcal{N}_1).$$

In this simple example it is easy to compute the admissible vector fields for $\mathcal{N}_1$ and $\mathcal{N}_2$ and to check that they are equivariant with respect to the representation in the sense of Definition 14.3. Nevertheless, consider an arbitrary collection of smooth vector fields on the total phase spaces of both networks $F = (F_1, F_2)\colon V^2 \to V^2$ and $G = G_1\colon V \to V$. Then equivariance with respect to the representation given by φ^* as in (14.2) yields

$$\varphi^* \circ F(x_1, x_2) = \varphi^* \begin{pmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{pmatrix} = F_1(x_1, x_2)$$
$$G \circ \varphi^*(x_1, x_2) = G_1(\varphi^*(x_1, x_2)) = G_1(x_1)$$

so that

$$F_1(x_1, x_2) = G_1(x_1).$$

In particular, the first component function of F does not depend on x_2. Summarizing, equivariance with respect to this quiver representation is equivalent to the fact that in the network $\mathcal{N}_1$ the state of cell 1 does not depend on that of cell 2. Here, this is essentially all the structural information one needs to describe the network $\mathcal{N}_1$, Once more, it is encoded in terms of generalized symmetry.

It remains to be seen how useful the usage of these algebraic objects turns out to be for the analysis of network dynamics. The two most pressing questions at the moment are

(i) Which properties of network structure can be encoded using graph fibrations and as a result in quiver representations?

(ii) Can the theory of equivariant dynamics be extended to quiver representations?

These questions are under investigation in the research project mentioned above. Regarding (i), the examples mentioned above are only a partial answer. In the publications that introduced graph fibrations to the network dynamics community (DeVille and Lerman [24, 25, 26]) one finds more examples of structural features that are encoded in different types of graph fibrations. Nevertheless, if the constructions presented in this chapter turn out to be useful for dynamical analysis a more thorough investigation appears to be necessary. In regard to the second question, it is desirable to determine dynamical properties that are inherent to a given structure. If that given structure is a quiver representation that encodes a structural feature of some network we would like to be able to determine behavior that is generic within the class of equivariant systems. As we have seen throughout this thesis, this requires dynamical systems machinery to respect generalized symmetries. As quiver representations are related to monoid representations, we are optimistic that we will be able to generalize some of the results – in particular regarding reduction methods – to this generalized setting. Nevertheless, this is beyond the scope of this thesis and shall be part of future research.

Part V

Appendix

Contents

Appendix A

Submanifolds of matrix spaces

We want to prove the results on submanifolds of the space $\mathrm{M}(n;\mathbb{K})$ of $n\times n$ matrices over $\mathbb{K}=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$ that were left out in Section 5.1. We treat $\mathrm{M}(n;\mathbb{K})$ as a real vector space meaning that we restrict scalar multiplication to real numbers. The result in the first proposition is well-known in the cases $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. The necessary arguments for the real case are sketched as an exercise in the equally known book Guillemin and Pollack [61]. We state them here especially for the case $\mathbb{K}=\mathbb{H}$ even though the same proof holds true for all three cases. For additional background on quaternionic matrices consult Zhang [117].

We begin with a brief summary of some facts and definitions that are particularly important for our considerations. Consider scalar multiplication by quaternions for a moment. As multiplication of quaternions is not commutative we have to distinguish between left and right scalar multiplication and *left* and *right linear (in)dependence* (over $\mathbb{H}$). The *rank* of a quaternionic matrix A is the number of right linear independent column vectors or equally the number of left linear independent row vectors of A. A quadratic matrix $A\in\mathrm{M}(n;\mathbb{H})$ is invertible (there exists $B\in\mathrm{M}(n;\mathbb{H})$ such that $AB=BA=\mathbb{1}$) if and only if it has full rank n. Furthermore, $\operatorname{rank}PAQ=\operatorname{rank}A$ for any invertible matrices P and Q of suitable dimensions.

Proposition A.1. *Let $\mathbb{K}=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$ and $\mathrm{M}(n;\mathbb{K})$ be the space of all $n\times n$ matrices with entries in $\mathbb{K}$ considered as a real vector space. Then*

$$\mathbf{M}_r(n;\mathbb{K})=\{A\in\mathrm{M}(n;\mathbb{K})\mid\operatorname{rank}A=r\}$$

with $r=0,\dots,n$ is a submanifold of codimension $(n-r)^2\dim\mathbb{K}$.

Proof. As $\mathbf{M}_0(n;\mathbb{K})=\{0\}$ and $\mathbf{M}_n(n;\mathbb{K})=\{A\in\mathrm{M}(n;\mathbb{K})\mid A\text{ invertible}\}$, the special cases $r=0$ and $r=n$ are clear. Hence, let $r\in\{1,\dots,n-1\}$ and $L\in\mathbf{M}_r(n;\mathbb{K})$. Then L has r (right) linear independent column vectors $v_1,\dots,v_r$. Without loss of generality (by exchanging columns of L) we may assume that these are the first r columns of L. The $n\times r$ matrix $(v_1,\dots,v_r)$ consisting of those column vectors still has rank r. Hence it has r (left) linear independent row vectors. Exchanging rows of that matrix allows us to assume that the first r rows are (left) linear independent. Applying the same exchange of rows to the full matrix L allows us to assume

$$L=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$$

where $A\in\mathrm{M}(r;\mathbb{K})$ is invertible – especially $\operatorname{rank}A=r$ – and $B\in\mathrm{M}(r\times(n-r);\mathbb{K})$, $C\in\mathrm{M}((n-r)\times r;\mathbb{K})$ and $D\in\mathrm{M}((n-r);\mathbb{K})$. Consider the matrix

$$L_0=\begin{pmatrix}\mathbb{1} & -A^{-1}B\\ 0 & \mathbb{1}\end{pmatrix}$$

where the dimensions of the identity matrices are suitably chosen. Then

$$LL_0 = \begin{pmatrix} A & 0 \\ C & -CA^{-1}B + D \end{pmatrix}.$$

As L_0 is clearly invertible

$$r = \operatorname{rank} L = \operatorname{rank} LL_0 = \operatorname{rank} A + \operatorname{rank}\left(-CA^{-1}B + D\right).$$

But already $\operatorname{rank} A = r$ and thus

$$\operatorname{rank}\left(-CA^{-1}B + D\right) = 0$$

which is only fulfilled by the zero matrix $0 \in \mathrm{M}\left((n-r);\mathbb{K}\right)$. Furthermore, these considerations show that any block matrix

$$K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

whose upper left block $\alpha \in \mathrm{M}\left(r;\mathbb{K}\right)$ is invertible has rank r if and only if

$$-\gamma\alpha^{-1}\beta + \delta = 0.$$

We may now choose a suitably small neighborhood U around L in a suitable topology such that every $K \in U$ is of the form

$$K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\alpha \in \mathrm{M}\left(r;\mathbb{K}\right)$ invertible. If K is close to L then α is close to A and hence is invertible as well. To see this in the quaternionic case we may use Theorem 7.3 in ZHANG [117] which dates back to WOLF [116] and connects the rank of a quaternionic matrix to that of its complex adjoint matrix. On the neighborhood U we define a map

$$\begin{aligned} f\colon U &\to \mathrm{M}\left((n-r);\mathbb{K}\right) \\ K &\mapsto -\gamma\alpha^{-1}\beta + \delta. \end{aligned}$$

This map is smooth and we have seen that $K \in U$ has rank r if and only if $f(K) = 0$.

We have to check that the derivative of f at $K = L$ is surjective on tangent spaces. As the target space of f is a linear space, its tangent space is the same space $\mathrm{M}\left((n-r);\mathbb{K}\right)$. To prove surjectivity let $X \in \mathrm{M}\left((n-r);\mathbb{K}\right)$ be arbitrary and consider the smooth curve

$$\nu(t) = L + t\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}$$

with t being restricted to an interval around 0 so that $\nu(t) \in U$ for all t. Then

$$f(\nu(t)) = -\gamma\alpha^{-1}\beta + \delta + tX$$

and

$$\frac{d}{dt} f(\nu(0)) = X.$$

This proves surjectivity of Df. Hence $\mathbf{M}_r\left(n;\mathbb{K}\right) \cap U = f^{-1}(0)$ is a submanifold of codimension

$$\operatorname{codim} \mathbf{M}_r\left(n;\mathbb{K}\right) = \dim \mathrm{M}\left((n-r);\mathbb{K}\right) = (n-r)^2 \dim \mathbb{K}$$

which completes the proof. □

The next Proposition treats nilpotent real matrices whose rank is reduced by one. Embedding them into all matrices of that rank would yield a codimension 1 submanifold using the last proposition. The proof relies on the normal form of matrices presented in ARNOLD [15].

Proposition A.2. *The collection of real nilpotent matrices of rank* $n-1$ *is a submanifold of* $\mathrm{M}(n;\mathbb{R})$ *of codimension* n*.*

Proof. Let $L \in \mathrm{M}(n;\mathbb{R})$ be nilpotent and $\operatorname{rank} L = n-1$. This directly yields that the Jordan normal form of L consists of precisely one Jordan block

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

In ARNOLD [15] a matrix normal form is presented that depends smoothly on the matrix – more precisely speaking a versal deformation. Any suitably small perturbation K of L is conjugate to a matrix

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ a_1 & \dots & \dots & a_n \end{pmatrix} \tag{A.1}$$

with $a_1, \dots, a_n \in \mathbb{R}$ depending smoothly on K. As K is an $n \times n$ matrix it is nilpotent if and only if $K^n = 0$. However, K^n is conjugate to

$$\begin{pmatrix} a_1 & \dots & \dots & a_n \\ \bullet & \dots & \dots & \bullet \\ \vdots & & & \vdots \\ \bullet & \dots & \dots & \bullet \end{pmatrix}$$

which can only be 0 if $a_i = 0$ for all $i = 1, \dots, n$. This yields that K close to L is nilpotent (and of rank $n-1$) if and only if it is conjugate to L. The deformation in (A.1) is constructed to have the minimal number of parameters which is n. It equals the codimension of the conjugacy orbit of L. Therefore the collection of matrices conjugate to L is a submanifold of $\mathrm{M}(n;\mathbb{R})$ of codimension n. $\square$

This appendix is nearly identical to Appendix A in our publication [97].

Appendix B

Computations for steady state bifurcations of feedforward networks

In this appendix we fill the gaps left in Chapter 10 by proving Lemmas 10.18 and 10.19. Recall that $\mathcal{L}_p = \{\sigma \in \Sigma \mid \sigma(p) = p\}$.

Lemma B.1 (Lemma 10.18)**.** *Let C be the set of cells of a feedforward network and $p \in C$ be non-maximal. Assume*

$$\sum_{\sigma \in \mathcal{L}_p} a_\sigma \neq 0$$

and

$$d_q = -\frac{\ell}{\sum_{\sigma \in \Sigma} a_\sigma},$$

for all $q \rhd p$. Then

$$-\frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} + \ell}{\sum_{\sigma \in \mathcal{L}_p} a_\sigma} = -\frac{\ell}{\sum_{\sigma \in \Sigma} a_\sigma}.$$

Proof. Let $p \in C$ be non-maximal. Assume

$$d_q = -\frac{\ell}{\sum_{\sigma \in \Sigma} a_\sigma}$$

for all $q \rhd p$. Then

$$\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} + \ell = -\left(\frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau}{\sum_{\sigma \in \Sigma} a_\sigma} - 1\right) \cdot \ell.$$

Furthermore,

$$\frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau}{\sum_{\sigma \in \Sigma} a_\sigma} - 1 = \frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau - \sum_{\sigma \in \Sigma} a_\sigma}{\sum_{\sigma \in \Sigma} a_\sigma} = -\frac{\sum_{\sigma \in \mathcal{L}_p} a_\sigma}{\sum_{\sigma \in \Sigma} a_\sigma}.$$

Hence, we obtain

$$-\frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau d_{\tau(p)} + \ell}{\sum_{\sigma \in \mathcal{L}_p} a_\sigma} = \frac{1}{\sum_{\sigma \in \mathcal{L}_p} a_\sigma} \cdot \left(\frac{\sum_{\tau \notin \mathcal{L}_p} a_\tau}{\sum_{\sigma \in \Sigma} a_\sigma} - 1\right) \cdot \ell = -\frac{\ell}{\sum_{\sigma \in \Sigma} a_\sigma}.$$

□

Lemma B.2 (Lemma 10.19). *Let C be the set of cells of a feedforward network and $p \in C$ be non-maximal. Assume*

$$\sum_{\sigma \in \mathcal{L}_p} a_\sigma = 0$$

and

$$d_q = -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}, \qquad R_q = -\frac{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}\ell^2 - \sum_{\sigma\in\Sigma} a_\sigma \sum_{\sigma\in\Sigma} f_{\sigma\lambda}\ell + \left(\sum_{\sigma\in\Sigma} a_\sigma\right)^2 f_{\lambda\lambda}}{\left(\sum_{\sigma\in\Sigma} a_\sigma\right)^3}$$

with $f_{\sigma\tau} = f_{\tau\sigma}$ for all $q \rhd p$. Define

$$\begin{aligned}
A &= \sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau},\\
B &= \sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda} + 2\sum_{\sigma\in\mathcal{L}_p,\tau\notin\mathcal{L}_p} f_{\sigma\tau} d_{\tau(p)},\\
C &= \sum_{\tau\notin\mathcal{L}_p} a_\tau R_{\tau(p)} + \sum_{\tau\notin\mathcal{L}_p} f_{\tau\lambda} d_{\tau(p)} + \sum_{\sigma,\tau\notin\mathcal{L}_p} f_{\sigma\tau} d_{\sigma(p)} d_{\tau(p)} + f_{\lambda\lambda},\\
K &= \sum_{\sigma\in\Sigma} a_\sigma,\\
E &= \sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda} - 2\cdot\frac{\ell}{K}\sum_{\substack{\sigma\in\Sigma\\ \tau\in\mathcal{L}_p}} f_{\sigma\tau}.
\end{aligned}$$

Then generically

$$B^2 - 4AC = E^2 > 0$$

and

$$\frac{-B+E}{2A} = -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}, \qquad \frac{-B-E}{2A} = M$$

with

$$M = \frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma}\cdot\left(1 + 2\frac{\sum_{\sigma\in\mathcal{L}_p,\tau\notin\mathcal{L}_p} f_{\sigma\tau}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}\right) - \frac{\sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda}}{\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}}.$$

Proof. Let $p \in C$ be non-maximal. Assume

$$d_q = -\frac{\ell}{\sum_{\sigma\in\Sigma} a_\sigma} \quad \text{and} \quad R_q = -\frac{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}\ell^2 - \sum_{\sigma\in\Sigma} a_\sigma \sum_{\sigma\in\Sigma} f_{\sigma\lambda}\ell + \left(\sum_{\sigma\in\Sigma} a_\sigma\right)^2 f_{\lambda\lambda}}{\left(\sum_{\sigma\in\Sigma} a_\sigma\right)^3}$$

for all $q \rhd p$. A key observation is

$$\sum_{\tau\notin\mathcal{L}_p} a_\tau = \sum_{\sigma\in\Sigma} a_\sigma,$$

as $\sum_{\sigma\in\mathcal{L}_p} a_\sigma = 0$. We denote this sum by K. Hence,

$$\begin{aligned}
\sum_{\tau\notin\mathcal{L}_p} a_\tau R_{\tau(p)} &= -\frac{\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau}\ell^2 - K\sum_{\sigma\in\Sigma} f_{\sigma\lambda}\ell + K^2 f_{\lambda\lambda}}{K^2}\\
&= -\frac{\ell^2}{K^2}\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau} + \frac{\ell}{K}\sum_{\sigma\in\Sigma} f_{\sigma\lambda} - f_{\lambda\lambda}.
\end{aligned}$$

Note that

$$\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau} = \sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau} + 2\sum_{\substack{\sigma\in\mathcal{L}_p\\ \tau\notin\mathcal{L}_p}} f_{\sigma\tau} + \sum_{\sigma,\tau\notin\mathcal{L}_p} f_{\sigma\tau},$$

where we have used $f_{\sigma\tau} = f_{\tau\sigma}$. Similar considerations occur frequently in the remainder of this proof. We use them without explicitly mentioning them. Furthermore, we compute

$$\begin{aligned}
\sum_{\tau\notin\mathcal{L}_p} f_{\tau\lambda} d_{\tau(p)} &= -\frac{\ell}{K}\sum_{\tau\notin\mathcal{L}_p} f_{\tau\lambda},\\
\sum_{\sigma,\tau\notin\mathcal{L}_p} f_{\sigma\tau} d_{\sigma(p)} d_{\tau(p)} &= \frac{\ell^2}{K^2}\sum_{\sigma,\tau\notin\mathcal{L}_p} f_{\sigma\tau}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
C &= \sum_{\tau\notin\mathcal{L}_p} a_\tau R_{\tau(p)} + \sum_{\tau\notin\mathcal{L}_p} f_{\tau\lambda} d_{\tau(p)} + \sum_{\sigma,\tau\notin\mathcal{L}_p} f_{\sigma\tau} d_{\sigma(p)} d_{\tau(p)} + f_{\lambda\lambda}\\
&= -\frac{\ell^2}{K^2}\sum_{\sigma,\tau\in\Sigma} f_{\sigma\tau} + \frac{\ell}{K}\sum_{\sigma\in\Sigma} f_{\sigma\lambda} - f_{\lambda\lambda} - \frac{\ell}{K}\sum_{\tau\notin\mathcal{L}_p} f_{\tau\lambda} + \frac{\ell^2}{K^2}\sum_{\sigma,\tau\notin\mathcal{L}_p} f_{\sigma\tau} + f_{\lambda\lambda}\\
&= -\frac{\ell^2}{K^2}\left(\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau} + 2\sum_{\substack{\sigma\in\mathcal{L}_p\\ \tau\notin\mathcal{L}_p}} f_{\sigma\tau}\right) + \frac{\ell}{K}\sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda}.
\end{aligned}$$

Next, we compute

$$\begin{aligned}
B^2 &= \left(\sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda} + 2\sum_{\substack{\sigma\in\mathcal{L}_p\\ \tau\notin\mathcal{L}_p}} f_{\sigma\tau} d_{\tau(p)}\right)^2\\
&= \left(\sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda} - 2\cdot\frac{\ell}{K}\sum_{\substack{\sigma\in\Sigma\\ \tau\in\mathcal{L}_p}} f_{\sigma\tau} + 2\cdot\frac{\ell}{K}\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}\right)^2\\
&= \left(\sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda} - 2\cdot\frac{\ell}{K}\sum_{\substack{\sigma\in\Sigma\\ \tau\in\mathcal{L}_p}} f_{\sigma\tau}\right)^2 + 4\cdot L
\end{aligned}$$

with

$$\begin{aligned}
L &= \frac{\ell}{K}\left(\sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda} - 2\cdot\frac{\ell}{K}\sum_{\substack{\sigma\in\Sigma\\ \tau\in\mathcal{L}_p}} f_{\sigma\tau}\right)\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau} + \frac{\ell^2}{K^2}\left(\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}\right)^2\\
&= \sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}\cdot\left(\frac{\ell}{K}\sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda} - \frac{\ell^2}{K^2}\left(2\cdot\sum_{\substack{\sigma\in\Sigma\\ \tau\in\mathcal{L}_p}} f_{\sigma\tau} - \sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}\right)\right)\\
&= \sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau}\cdot\left(\frac{\ell}{K}\sum_{\sigma\in\mathcal{L}_p} f_{\sigma\lambda} - \frac{\ell^2}{K^2}\left(\sum_{\sigma,\tau\in\mathcal{L}_p} f_{\sigma\tau} + 2\cdot\sum_{\substack{\sigma\in\mathcal{L}_p\\ \tau\notin\mathcal{L}_p}} f_{\sigma\tau}\right)\right) = A\cdot C.
\end{aligned}$$

Hence,

$$
\begin{aligned}
B^2 - 4AC &= \left(\sum_{\sigma \in \mathcal{L}_p} f_{\sigma\lambda} - 2 \cdot \frac{\ell}{K} \sum_{\substack{\sigma \in \Sigma \\ \tau \in \mathcal{L}_p}} f_{\sigma\tau} \right)^2 + 4L - 4AC \\
&= \left(\sum_{\sigma \in \mathcal{L}_p} f_{\sigma\lambda} - 2 \cdot \frac{\ell}{K} \sum_{\substack{\sigma \in \Sigma \\ \tau \in \mathcal{L}_p}} f_{\sigma\tau} \right)^2 \\
&= E^2.
\end{aligned}
$$

Generically, this expression is positive, which allows us to compute

$$
\begin{aligned}
-B + E &= 2 \cdot \frac{\ell}{K} \left(\sum_{\substack{\sigma \in \mathcal{L}_p \\ \tau \notin \mathcal{L}_p}} f_{\sigma\tau} - \sum_{\substack{\sigma \in \Sigma \\ \tau \in \mathcal{L}_p}} f_{\sigma\tau} \right) \\
&= -2 \cdot \frac{\ell}{K} \sum_{\sigma, \tau \in \mathcal{L}_p} f_{\sigma\tau} = -2A \cdot \frac{\ell}{K},
\end{aligned}
$$

proving

$$
\frac{-B + E}{2A} = -\frac{\ell}{\sum_{\sigma \in \Sigma} a_\sigma}.
$$

On the other hand

$$
\begin{aligned}
-B - E &= - \sum_{\sigma \in \mathcal{L}_p} f_{\sigma\lambda} + 2 \cdot \frac{\ell}{K} \sum_{\substack{\sigma \in \mathcal{L}_p \\ \tau \notin \mathcal{L}_p}} f_{\sigma\tau} - \left(\sum_{\sigma \in \mathcal{L}_p} f_{\sigma\lambda} - 2 \cdot \frac{\ell}{K} \sum_{\substack{\sigma \in \Sigma \\ \tau \in \mathcal{L}_p}} f_{\sigma\tau} \right) \\
&= 2 \cdot \left(\frac{\ell}{K} \left(\sum_{\sigma, \tau \in \mathcal{L}_p} f_{\sigma\tau} + 2 \sum_{\substack{\sigma \in \mathcal{L}_p \\ \tau \notin \mathcal{L}_p}} f_{\sigma\tau} \right) - \sum_{\sigma \in \mathcal{L}_p} f_{\sigma\lambda} \right),
\end{aligned}
$$

proving

$$
\frac{-B - E}{2A} = \frac{\ell}{\sum_{\sigma \in \Sigma} a_\sigma} \cdot \left(1 + 2 \frac{\sum_{\sigma \in \mathcal{L}_p, \tau \notin \mathcal{L}_p} f_{\sigma\tau}}{\sum_{\sigma, \tau \in \mathcal{L}_p} f_{\sigma\tau}} \right) - \frac{\sum_{\sigma \in \mathcal{L}_p} f_{\sigma\lambda}}{\sum_{\sigma, \tau \in \mathcal{L}_p} f_{\sigma\tau}} = M.
$$

$\square$

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List of Figures

List of Symbols

N	The number of cells in a network
$\mathbf{D}_N$	The dihedral group
$\mathbf{S}_N$	The symmetric group
$\mathbb{Z}_4$	The cyclic group
$\mathcal{N}$	A network
C	The set of nodes or cells
E	The set of edges or arrows
$\sim_C$	Type-equivalence relation of cells
$\sim_E$	Type/Color-equivalence relation of arrows
$H(e)$	The head of an arrow
$T(e)$	The tail of an arrow
$I(p)$	The input set / input tree of cell p
$\sim_I$	Input equivalence of cells
$B(p,q)$	The set of input isomorphisms from p to q
$\mathcal{B}_{\mathcal{N}}$	The (symmetry) groupoid of $\mathcal{N}$
V_p	The internal phase space of cell p in a non-homogeneous network
$\prod_{p\in C} V_p$	The total phase space of a non-homogeneous network
$V_{T(I(p))}$	The coupling phase space of p
$\bowtie$	A (balanced) coloring
$\Delta_{\bowtie}$	A polysynchronous subspace corresponding to the coloring $\bowtie$
$\bowtie_y$	The (balanced) coloring associated to the hyperbolic equilibrium y
$\mathcal{N}_{\bowtie}$	The quotient network with respect to $\bowtie$
$C_{\bowtie}$	The set of cells of the quotient network $\mathcal{N}_{\bowtie}$
$E_{\bowtie}$	The set of edges or arrows in the quotient network $\mathcal{N}_{\bowtie}$
$\mathfrak{C}$	A set of cell classes
Graph	The category of networks and maps of networks
$\mathcal{P}$	The map assigning a manifold as internal phase space
Man	The category of smooth manifolds and smooth maps
Graph/Man	The category of networks of manifolds and maps of networks of manifolds
$\prod_{p\in C} \mathcal{P}(p)$	The categorical product of internal phase spaces
$\mathbb{P}\mathcal{N}$	The total phase space of $\mathcal{N}$
Control$(M \times U \to TM)$	The vector space of open systems
C_p	The set of of cells of the input tree $I(p)$
E_p	The set of arrows of the input tree $I(p)$
$\mathfrak{Ctrl}(\mathcal{N}, \mathcal{P})$	The collections of open systems on the input trees
$\mathfrak{I}$	The interconnection map
$\mathrm{im}\,(L)$	The image of L
$\mathcal{W}_{\mathcal{C}}$	The symmetric monoidal category of $\mathcal{C}$-labeled boxes and wiring diagrams
$\widetilde{\textbf{Int}}$	The category of continuous interval sheaves.
Σ	The semigroup of input maps
$\omega_{p,q}$	A path from p to q
$\Omega_{p,q}$	The set of all paths from p to q without loops
$\mathcal{N}_p$	The input network of p

C_p	The set of cells of the input network $\mathcal{N}_p$
$\bigoplus_{p\in C} V$	The total phase space of a homogeneous coupled cell system
$\bigoplus_{\sigma\in\Sigma} V$	The coupling phase space of a homogeneous coupled cell system The total phase space of the fundamental network
$\mathcal{C}^\infty$	The space of smooth functions
γ_f	The homogeneous coupled cell vector field with internal dynamics f
B_σ	The adjacency matrix of the arrow color σ / The generalized basis for the endomorphisms of the right regular representation
$\mathbb{1}_V$	The identity on the internal phase space V
$\mathfrak{gl}(V)$	The space of linear maps on the internal phase space V
$\mathbf{p}$	A symmetry of a network
$\mathbf{P_p}$	The representation map corresponding to the network symmetry $\mathbf{p}$
Δ_P	The synchrony subspace induced by the partition P
φ^*	The linear semiconjugacy induced by the graph fibration φ
Δ_φ	The robust synchrony subspace induced by the graph fibration φ
$\langle\Sigma\rangle$	The semigroup generated by Σ
Id_C	The identity input map
Id	The identity element in the monoid Σ
$\widetilde{\mathcal{N}}$	The fundamental network of $\mathcal{N}$
Γ_f	The fundamental network vector field with internal dynamics f
$\widetilde{\Sigma}$	The monoid of input maps of the fundamental network
$\widetilde{\sigma}$	The input map of the fundamental network induced by σ
Δ_p	The robust synchrony subspace in the fundamental network that is the input network of p
A_σ	The right regular representation map of σ
f_λ	A family of response functions
$\gamma_{f_\lambda,\lambda}$	A family of admissible vector fields
spec	The spectrum of a linear map
$\mathbf{i}$	The imaginary unit
$\circ_\Sigma$	Concatenation of response functions in semigroup networks
$[\cdot,\cdot]_\Sigma$	Lie bracket of response functions in semigroup networks
$\mathcal{P}^{i,j}$	The homogeneous polynomials of degree $i+1$ in X and j in λ
ad_f^Σ	The function mapping a smooth response function to its Lie bracket with f
$\mathbf{V}$	A finite-dimensional real vector space / monoid representation
Θ	An arbitrary monoid
T_θ	The representation matrix of an element $\theta\in\Theta$
$\mathbb{E}_\mu$	Generalized eigenspace to the eigenvalue μ
$\ker_0(L)$	The generalized kernel of L
$\mathrm{im}_0(L)$	The reduced image of L
P^c	The projection onto the generalized kernel / center subspace
P^h	The projection onto the reduced image / hyperbolic subspace
X^c	The $\ker_0(L)$- / $\mathcal{X}^c$-component of X
X^h	The $\mathrm{im}_0(L)$ / $\mathcal{X}^h$-component of X
$\mathcal{X}^c$	The center subspace
$\mathcal{X}^h$	The hyperbolic subspace
$\underline{X}$	Coordinates for the extended system
$\underline{\Gamma}_f$	The extended fundamental network vector field
$\underline{A}_\sigma$	The extended monoid representation
$\underline{\mathcal{X}}^c$	The center subspace of the extended system
$\underline{\mathcal{X}}^h$	The hyperbolic subspace of the extended system
$\underline{P}^c$	The projection onto the generalized kernel/center subspace of the extended system
$\underline{P}^h$	The projection onto the hyperbolic subspace of the extended system
$\underline{X}^c$	The $\underline{\mathcal{X}}^c$-component of $\underline{X}$
$\underline{X}^h$	The $\underline{\mathcal{X}}^h$-component of $\underline{X}$

P' — The bijective linear map $\underline{\mathcal{X}}^c \to \mathcal{X}^c \times \mathbb{R}^l$

M^c — The center manifold

$\underline{\psi}$ — The map whose graph is the center manifold M^c

Q' — The inverse of P'

$\mathrm{Hom}_\Theta(\mathbf{V}, \mathbf{V}')$ — The space of homomorphisms of Θ-representations $\mathbf{V}$ and $\mathbf{V}'$

$\mathrm{End}_\Theta(\mathbf{V})$ — The algebra of endomorphisms of the Θ-representation $\mathbf{V}$

$\mathrm{End}_\Theta^{\mathrm{nil}}(\mathbf{V})$ — The nilpotent endomorphisms of the Θ-representation $\mathbf{V}$

$\mathrm{ind}\,(\mathbf{V})$ — The index of the indecomposable representation $\mathbf{V}$ of Θ

$\dim \mathbf{V}$ — The dimension of the vector space $\mathbf{V}$

$\mathbb{R}\Sigma$ — The monoid algebra

$\delta_{\tau,\sigma}$ — The Kronecker delta

$\mathbb{R}\Sigma_{\mathbb{R}\Sigma}$ — The right regular module

$\mathbb{R}\Sigma^{\mathrm{op}}$ — The opposite monoid algebra

$\mathrm{M}\,(s;\mathbb{K})$ — The algebra of $s \times s$ matrices with entries in $\mathbb{K}$

$\mathrm{End}_\Theta^{\mathrm{b-nil}}(\mathbf{V})$ — The endomorphisms of an isotypic component whose matrix representation has only nilpotent entries

$\mathrm{tr}\,(L)$ — The trace of L

$\mathrm{rank}\, L$ — The rank of L

$\mathrm{M}_i\,(s;\mathbb{K})$ — The submanifold of $s \times s$ matrices with rank $s-i$

$\mathrm{M}_1^{\mathrm{nil}}\,(s;\mathbb{R})$ — The real nilpotent $s \times s$ matrices of rank $s-1$

$\mathrm{Iso}(\mathbf{X})$ — The subset of endomorphisms whose generalized kernel is isomorphic to $\mathbf{X}$

$\lceil \cdot \rceil$ — The ceiling function

1D — Abbreviation of the case of one-dimensional internal dynamics

DD — Abbreviation of the case of high-dimensional internal dynamics

W — A high-dimensional internal phase space

$\mathbf{V}^1$ — The total phase space of the fundamental network / right regular representation space in the case **1D**

$\mathbf{V}^{\mathsf{D}}$ — The total phase space of the fundamental network / right regular representation space in the case **DD**

A_σ^1 — The right regular representation maps in the case **1D**

A_σ^{D} — The right regular representation maps in the case **DD**

B_σ^1 — The adjacency matrix of the arrow color σ in the case **1D**

B_σ^{D} — The adjacency matrix of the arrow color σ in the case **DD**

$\lfloor \cdot \rfloor$ — The floor function

$\mathcal{X}_1^c$ — The generalized kernel / center subspace in the case **1D**

P_1^c — The projection onto the generalized kernel / center subspace in the case **1D**

$\mathcal{X}_{\mathsf{D}}^c$ — The generalized kernel / center subspace in the case **DD**

$\underline{\psi}^{\mathsf{D}}$ — The map whose graph is the center manifold M^c in the case **DD**

Q'_{D} — The inverse of P' in the case **DD**

$\multimap\!\!\circ$ — The loop-type equivalence relation

$\mathcal{L}_p$ — The subset of input maps that denote self-loops of p

$\trianglelefteq$ — The preorder in a network induced by input relations

$\triangleleft$ — The strict preorder relation corresponding to $\trianglelefteq$

$\vartriangle$ — The equivalence relation induced by the preorder $\trianglelefteq$

$\theta L \theta'$ — Green's L-relation on monoid elements

$\leq_L$ — Green's-L-preorder on monoid elements

a_σ — The Taylor coefficient of the response function f in direction of its σ-input

ℓ — The Taylor coefficient of the response function f in the direction of the parameter λ

$f_{\sigma\tau}, f_{\sigma\lambda}, f_{\lambda\lambda}$ — The second order Taylor coefficients of the response function f

μ_p — The square root order of cell p in a generic branch of steady states in a feedforward network

Γ_ϕ^1	The fundamental network vector field with internal dynamics ϕ in the case **1D**
Γ_f^{D}	The fundamental network vector field with internal dynamics f in the case **DD**
$\mathcal{Q}_0$	A quiver
$\mathcal{Q}_0$	The set of vertices of $\mathcal{Q}$
$\mathcal{Q}_1$	The set of arrows of $\mathcal{Q}$

Anhang

Summary

This thesis deals with the investigation of dynamical properties – in particular generic synchrony breaking bifurcations – that are inherent to the structure of a semigroup network as well the numerous algebraic structures that are related to these types of networks. Most notably we investigate the interplay between network dynamics and monoid representation theory as induced by the fundamental network construction in terms of hidden symmetry as introduced in RINK and SANDERS [92].

After providing a brief survey of the field of network dynamics in Part I, we thoroughly introduce the formalism of semigroup networks, the customized dynamical systems theory, and the necessary background from monoid representation theory in Chapters 3 and 4. The remainder of Part II investigates generic synchrony breaking bifurcations and contains three major results. The first is Theorem 5.11, which shows that generic symmetry breaking steady state bifurcations in monoid equivariant dynamics occur along absolutely indecomposable subrepresentations – a natural generalization of the corresponding statement for group equivariant dynamics. Then Theorem 7.12 relates the decomposition of a representation given by a network with high-dimensional internal phase spaces to that induced by the same network with one-dimensional internal phase spaces. This result is used to show that there is a smallest dimension of internal dynamics in which all generic l-parameter bifurcations of a fundamental network can be observed (Theorem 7.24).

In Part III, we employ the machinery that was summarized and further developed in Part II to feedforward networks. We propose a general definition of this structural feature of a network and show that it can equivalently be characterized in different algebraic notions in Theorem 8.35. These are then exploited to fully classify the corresponding monoid representation for any feedforward network and to classify generic synchrony breaking steady state bifurcations with one- or high-dimensional internal dynamics.

Zusammenfassung

Die vorliegende Dissertationsschrift analysiert dynamisches Verhalten – insbesondere generische Bifurkationen von synchronen Ruhelagen – von dynamischen Systemen mit der zugrundeliegenden Struktur eines Halbgruppennetzwerks (semigroup network), die von der Netzwerkstruktur diktiert werden. Ein Schwerpunkt liegt dabei auf der Untersuchung des Zusammenspiels von Netzwerkdynamik und der Darstellungstheorie von Monoiden, welche über die Konstruktion eines Fundamentalnetzwerks und versteckte Symmetrien Eintritt erhält (siehe RINK und SANDERS [92]).

Nachdem Teil I einen Überblick über die Entwicklung der Netzwerkdynamik und der dafür entwickelten Zugänge gibt, stellen wir Halbgruppennetzwerke, die darauf angepasste klassische Theorie der dynamischen Systeme sowie die notwendigen Grundlagen aus der Darstellungstheorie von Monoiden vor (Kapitel 3 und 4). Darüberhinaus untersuchen wir in Teil II generische Bifurkationen von synchronen Ruhelagen. Dieser Teil enthält drei Hauptresultate. Zunächst beweist Theorem 5.11, dass symmetriebrechende Bifurkationen in Systemen, die äquivariant bezüglich der Darstellung eines Monoiden sind, generischerweise entlang von absolut unzerlegbaren Unterdarstellungen auftreten – dies stellt eine natürliche Verallgemeinerung des zugehörigen Resultats für Gruppendarstellungen dar. Außerdem verbinden wir in Theorem 7.12 die Zerlegung der Darstellung, die von einem Netzwerk mit hochdimensionaler interner Dynamik erzeugt werden, mit der, die vom selben Netzwerk mit eindimensionaler interner Dynamik erzeugt wird. Dieses Resultat verwenden wir, um zu beweisen, dass es eine kleinstmögliche Dimension interner Dynamik gibt, in der alle generischen l-Parameter Bifurkationen eines Fundamentalnetzwerks auftreten (Theorem 7.24).

In Teil III wenden wir die Methoden und Techniken, die in Teil II vorgestellt und weiterentwickelt wurden, auf Netzwerke mit feedforward Struktur an. Wir definieren diese strukturelle Eigenschaft von Netzwerken allgemein und zeigen, dass sie äquivalent durch verschiedene algebraische Definitionen dargestellt werden kann (Theorem 8.35). Anschließend nutzen wir diese Darstellungen, um die zugehörige Monoiddarstellung eines beliebigen Netzwerks mit feedforward Struktur vollständig zu charakterisieren sowie generischerweise auftretenden Bifurkationen von synchronen Ruhelagen zu klassifizieren, sowohl im Fall eindimensionaler wie auch im Fall hochdimensionaler interner Dynamik.

Veröffentlichungen

Meine Arbeit an dem Dissertationsprojekt „Genericity in network dynamics“ unter Betreuung von Prof. Dr. Reiner Lauterbach am Fachbereich Mathematik der Universität Hamburg hat zu den folgenden Veröffentlichungen und Preprints geführt, deren Resultate in diese Dissertationsschrift eingeflossen sind.

(a) [97] S. SCHWENKER. „Generic Steady State Bifurcations in Monoid Equivariant Dynamics with Applications in Homogeneous Coupled Cell Systems“. *SIAM Journal on Mathematical Analysis* 50.3 (2018), S. 2466–2485

(b) [84] E. NIJHOLT, B. RINK und S. SCHWENKER. „Structural Properties and Bifurcations of Generalized Feedforward Networks“. Preprint (2019)

(c) [83] E. NIJHOLT, B. RINK und S. SCHWENKER. „Homogeneous Coupled Cell Systems with High-dimensional Internal Dynamics“. Preprint (2019)

(d) [82] E. NIJHOLT, B. RINK und S. SCHWENKER. „A new algorithm for computing idempotents of R-trivial monoids“ (2019). arXiv: 1906.02844
(Inhalt nicht Teil der Dissertationsschrift, siehe Remark 9.9)

Das Hauptresultat aus (a) wurde etwa zeitgleich im Frühjahr 2017 in NIJHOLT und RINK [77] signifikant verallgmeinert. Zudem sind die Autoren von [77] auch die Koautoren von (b) - (d). Nichtsdestotrotz ist (a) unabhängig von [77] entstanden. Eine Kooperation bestand zu diesem Zeitpunkt noch nicht. Die Resultate aus (b) - (d) sind in gemeinsam in stetigem Austausch entstanden. Kein Teil kann einem einzelnen der Autoren zugeordnet werden und das intellektuelle Eigentum gebührt allen gleichermaßen.

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den 25. September 2019