

Higher order interactions lead to “reluctant” synchrony breaking

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Abstract

To model dynamical systems on networks with higher order (non-pairwise) interactions, we recently introduced a new class of ODEs on hypernetworks. Here we consider one-parameter synchrony breaking bifurcations in such ODEs. We call a synchrony breaking steady state branch “reluctant” if it is tangent to a synchrony space, but does not lie inside it. We prove that reluctant synchrony breaking is ubiquitous in hypernetwork systems, by constructing a large class of examples that support it. We also give an explicit formula for the order of tangency to the synchrony space of a reluctant steady state branch.

1 Introduction

Recent advances in a large variety of research fields have highlighted the importance of non-pairwise interactions for the collective dynamical behavior of complex network systems. These so-called *higher order interactions* turn out to be crucial in problems from, e.g., neuroscience (see [1]), social science (see [2]), and ecology (see [3]). Higher order interaction networks have found their way into various recent mathematical studies as well. We mention in particular the theoretical papers [4, 5, 6, 7, 8, 9, 10], which investigate synchronisation in classes of networks with non-pairwise, nonlinear interaction in their equation of motion. We also refer to the excellent surveys [11, 12, 13, 14, 15] and references therein, for an in-depth discussion of higher order networks, and numerous examples of higher order network systems arising in applications.

This paper builds on previous work of the authors [10], in which we generalised the notion of a *coupled cell network*, introduced by Golubitsky, Stewart, Field et al. [16, 17, 18], to the context of higher order networks. We did this by introducing a class of “hypernetworks” and defining their “admissible” maps

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and ODEs, thus formalising the notion of a dynamical system on a higher order interaction network. We also introduced balanced colorings [16] of hypernetworks, and hypergraph fibrations [19, 20], and used these concepts to classify the *robust synchrony patterns*, that is, the synchrony spaces that are invariant under every admissible map, to hypernetwork dynamical systems.

Perhaps the most surprising result in [10] is the observation that the robust synchrony spaces of a hypernetwork system are not determined by linear terms in its equations of motion. This distinguishes hypernetworks from classical (dyadic) coupled cell networks, for which it was proved in [21] that a synchrony space is invariant under every admissible map, if and only if it is invariant under every linear admissible map, see also [22]. On the contrary, we prove in [10] that a synchrony space of a hypernetwork system is robustly invariant, whenever it is invariant under all polynomial admissible maps of a specific degree, which depends on the order of the hyperedges in the hypernetwork. Examples moreover show that our estimate for this polynomial degree is sharp.

As a consequence, a hypernetwork-admissible map of sufficiently low polynomial degree may admit “ghost” synchrony spaces that are not supported by general, e.g., higher degree polynomial admissible maps. These ghost synchrony spaces may have a profound effect on the dynamics of the hypernetwork system. In particular, the final section of [10] presents numerical evidence that they can give rise to a remarkable new type of synchrony breaking bifurcation.

We in fact observed this type of bifurcation in a one-parameter family of admissible ODEs for the hypernetwork depicted in Figure 2, meaning that these ODEs are of the form given in Equation (2.3) below. Figure 1 displays two numerically obtained branches of steady states that emerge in a bifurcation in a particular system of this form. The steady state branches were found by forward integrating the equations of motion – so they are asymptotically stable. We see that $y_0 = y_1$ for negative values of the bifurcation parameter λ , so on the negative branch y_0 and y_1 are synchronous. On the positive branch, y_0 and y_1 are nonsynchronous, i.e., for positive values of λ it holds that $y_0 \neq y_1$. However, the difference $y_0 - y_1$ only increases very slowly as a function of λ . In other words: the branch is tangent to the synchrony space $\{y_0 = y_1\}$. In [10], we called this phenomenon “reluctant synchrony breaking”. A more detailed numerical analysis, see Figures 3a and 3b, suggests that $y_1 - y_0 \sim \lambda^3$, i.e., that the branch has a third order tangency to the synchrony space. In Section 4, we will prove that this is indeed the case.

The goal of this paper is to show that reluctant synchrony breaking is ubiquitous in hypernetworks. The main result that we present is Theorem 4.2, which states that reluctant synchrony breaking occurs generically in one-parameter bifurcations in a large class of hypernetworks. These so-called *augmented hypernetworks* are constructed by coupling new nodes to an existing network or hypernetwork by means of specific higher order interactions. The hypernetwork depicted in Figure 2 is just one example of such an augmented hypernetwork. This means that the anomalous bifurcation that was discovered in [10] and described above is not a numerical artefact. Instead, reluctant synchrony breaking is a generic phenomenon in ODEs of the form (2.3). To illustrate our main result,

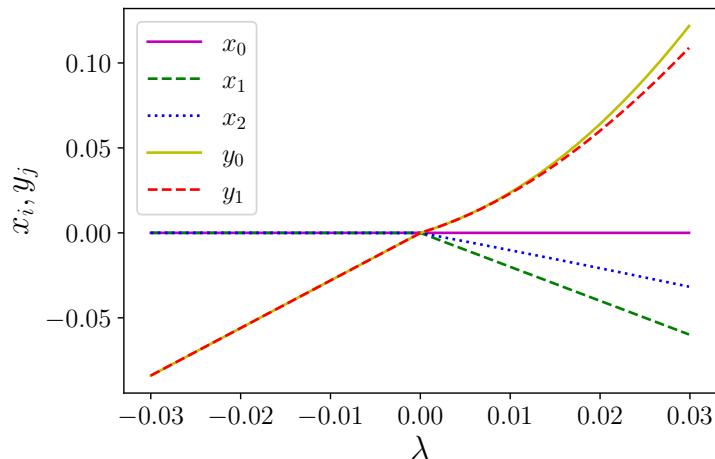


Figure 1: Numerically obtained steady state branches that emerge in a synchrony breaking bifurcation in an admissible ODE for the hypernetwork depicted in Figure 2. For $\lambda < 0$ the branch satisfies $y_0 = y_1$. We will prove that $y_1 - y_0 \sim \lambda^3$ for $\lambda > 0$. Figure taken from [10].

we present several more examples in this paper. We also argue (see Remark 5) that one may design augmented hypernetworks which admit reluctant synchrony breaking bifurcation branches with an arbitrarily high order of reluctance, i.e., an arbitrarily high order of tangency to a synchrony space.

Structure of the article. In Section 2, we illustrate reluctant synchrony breaking by studying the example presented in [10] in more detail. In Section 3, we recall our general mathematical framework for dynamical systems defined on hypernetworks, and we give the definition of an augmented hypernetwork. In Section 4, we prove our main result, Theorem 4.2, which states that reluctant synchrony breaking occurs generically in augmented hypernetworks, and which provides a formula for the order of reluctance of the synchrony breaking steady state branch. We also apply the theorem to the example discussed in this introduction. In Section 5, the main theorem is illustrated by three more examples.

2 A first example

We now provide more details on the example that was briefly discussed in the introduction, and that was introduced and studied numerically in [10]. As mentioned above, this example concerns the hypernetwork shown in Figure 2. In [10], we introduced the class of *admissible ODEs* of a hypernetwork (see also Section 3). The admissible ODEs associated to the hypernetwork in Figure 2

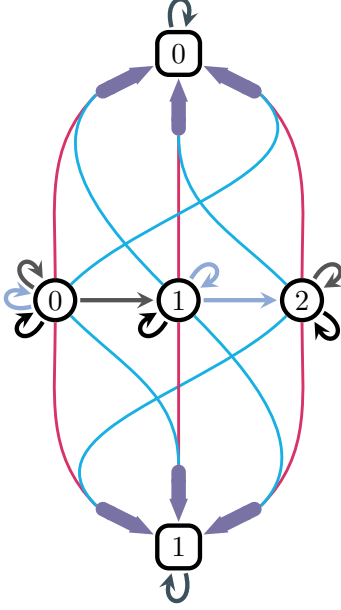


Figure 2: The hypernetwork that supports the unusual “reluctant” synchrony breaking steady-state branch shown in Figure 1.

are all the ODEs of the form

$$\begin{aligned}
 \dot{x}_0 &= G(x_0, x_0, x_0), \\
 \dot{x}_1 &= G(x_1, x_1, x_0), \\
 \dot{x}_2 &= G(x_2, x_1, x_2), \\
 \dot{y}_0 &= F(y_0, (\mathbf{x}_0, \mathbf{x}_1), (\mathbf{x}_1, \mathbf{x}_2), (\mathbf{x}_2, \mathbf{x}_0)), \\
 \dot{y}_1 &= F(y_1, (\mathbf{x}_0, \mathbf{x}_2), (\mathbf{x}_1, \mathbf{x}_0), (\mathbf{x}_2, \mathbf{x}_1)),
 \end{aligned} \tag{2.1}$$

for certain smooth functions F and G . We assume for now that the variables x_i, y_j take values in \mathbb{R} , so that $F : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. The brackets in F serve to distinguish the two-dimensional inputs from hyperedges of order two (the purple arrows in the figure.) The assumption that these hyperedges are identical, translates into the requirement that F is invariant under all permutations of these pairs of variables. That is, we require that for all $Y, X_1, \dots, X_6 \in \mathbb{R}$,

$$\begin{aligned}
 F(Y, (X_0, X_1), (X_2, X_3), (X_4, X_5)) &= \\
 F(Y, (X_2, X_3), (X_0, X_1), (X_4, X_5)) &= \\
 F(Y, (X_0, X_1), (X_4, X_5), (X_2, X_3)) &.
 \end{aligned} \tag{2.2}$$

To study bifurcations within this class of admissible ODEs, we parameterize the response functions F and G in (2.1) by a scalar variable λ taking values in some

open neighborhood $\Omega \subseteq \mathbb{R}$ of the origin. This gives the one-parameter family of hypernetwork admissible ODEs

$$\begin{aligned}
\dot{x}_0 &= G(x_0, x_0, x_0; \lambda), \\
\dot{x}_1 &= G(x_1, x_1, x_0; \lambda), \\
\dot{x}_2 &= G(x_2, x_1, x_2; \lambda), \\
\dot{y}_0 &= F(y_0, (x_0, x_1), (x_1, x_2), (x_2, x_0); \lambda), \\
\dot{y}_1 &= F(y_1, (x_0, x_2), (x_1, x_0), (x_2, x_1); \lambda).
\end{aligned} \tag{2.3}$$

To guarantee that the system (2.3) is admissible for every fixed value of $\lambda \in \Omega$, we assume F satisfies Equation (2.2) for any fixed value of λ . Let us in addition assume

$$F(0, (0, 0), (0, 0), (0, 0); 0) = G(0, 0, 0; 0) = 0,$$

so that the system (2.3) has a steady-state point at the origin for $\lambda = 0$. We may study the persistence of this steady-state by investigating the Jacobian at the origin of the system for $\lambda = 0$. To this end, we write

$$\begin{aligned}
&F(Y, (X_0, X_1), (X_2, X_3), (X_4, X_5); \lambda) \\
&= aY + bX_0 + cX_1 + bX_2 + cX_3 + bX_4 + cX_5 + d\lambda \\
&+ \mathcal{O}(\|(Y, X_0, \dots, X_5; \lambda)\|^2),
\end{aligned} \tag{2.4}$$

and

$$G(X_0, X_1, X_2; \lambda) = AX_0 + BX_1 + CX_2 + \mathcal{O}(|\lambda| + \|(X_0, X_1, X_2)\|^2), \tag{2.5}$$

with $a, \dots, d, A, B, C \in \mathbb{R}$, to specify the linear terms. The multiple occurrence of the terms b and c in (2.4) is due to the invariance properties of F in (2.2). In terms of these coefficients, the Jacobian matrix of the right hand side of Equation (2.3) at $(x, \lambda) = (0, 0)$ with respect to the spatial variables x is

$$\begin{pmatrix}
A + B + C & 0 & 0 & 0 & 0 \\
C & A + B & 0 & 0 & 0 \\
0 & B & A + C & 0 & 0 \\
b + c & b + c & b + c & a & 0 \\
b + c & b + c & b + c & 0 & a
\end{pmatrix}.$$

The eigenvalues of this Jacobian are $A + B + C$, $A + B$ and $A + C$ (all with multiplicity 1), and a (with geometric multiplicity 2). To allow for a steady state bifurcation to occur at $\lambda = 0$, we consider the case $A + B = 0$. We moreover assume the generic conditions $a, A + B + C, A + C \neq 0$ to hold.

We claim that as λ is varied near 0, two branches of steady states will generically emerge from the origin. These can be found by first focusing on the subnetwork given by the three nodes of the same type. That is, we first solve

$$\begin{aligned}
G(x_0, x_0, x_0; \lambda) &= 0, \\
G(x_1, x_1, x_0; \lambda) &= 0, \\
G(x_2, x_1, x_2; \lambda) &= 0.
\end{aligned}$$

A direct calculation shows that, for generic values of the first and second degree Taylor coefficients of G , one of the steady state branches is locally given by

$$x_0(\lambda) = x_1(\lambda) = x_2(\lambda) = x(\lambda) = D_0\lambda + \mathcal{O}(|\lambda|^2), \quad (2.6)$$

while another branch is given by

$$\begin{aligned} x_0(\lambda) &= D_0\lambda + \mathcal{O}(|\lambda|^2), & x_1(\lambda) &= D_1\lambda + \mathcal{O}(|\lambda|^2), \\ x_2(\lambda) &= D_2\lambda + \mathcal{O}(|\lambda|^2), \end{aligned} \quad (2.7)$$

for certain non-zero and mutually distinct $D_0, D_1, D_2 \in \mathbb{R}$. We omit the computation of these branches. For a detailed exposition on how to compute steady state bifurcation branches in so-called *feedforward networks* (i.e. networks with no loops other than self-loops), we refer to [23].

We now turn to computing the values of y_0 and y_1 along the bifurcation branches. We start by looking at the first branch, given by Equation (2.6). Restricted to this branch, the steady state equation $\dot{y}_0 = 0$ becomes

$$F(y_0, (x(\lambda), x(\lambda)), (x(\lambda), x(\lambda)), (x(\lambda), x(\lambda)); \lambda) = 0. \quad (2.8)$$

Combining (2.4) and (2.6) this can be expanded as

$$ay_0 + (3D_0(b+c) + d)\lambda + \mathcal{O}(\|(y_0; \lambda)\|^2) = 0,$$

which by the implicit function theorem has a unique solution given by

$$y_0(\lambda) = \frac{-3D_0(b+c) - d}{a}\lambda + \mathcal{O}(|\lambda|^2).$$

Setting $\dot{y}_1 = 0$ gives precisely the same equation to solve as (2.8), but with y_0 replaced by y_1 . Hence, we find $y_0(\lambda) = y_1(\lambda)$ along this first branch, which we will therefore refer to as the *synchronous branch* of system (2.3).

We now turn to the second branch of steady states, of which the asymptotics of the x -variables is given by Equation (2.7). Combining (2.4) with (2.7), we find that $\dot{y}_0 = 0$ is equivalent to

$$ay_0 + ((b+c)(D_0 + D_1 + D_2) + d)\lambda + \mathcal{O}(\|(y_0; \lambda)\|^2) = 0.$$

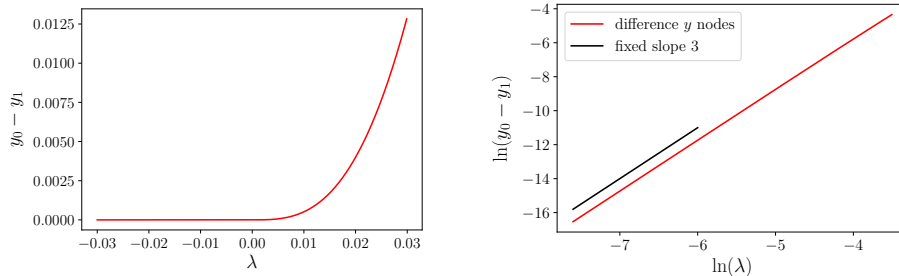
It follows again from the implicit function theorem that locally precisely one solution exists, given by

$$y_0(\lambda) = \frac{-(b+c)(D_0 + D_1 + D_2) - d}{a}\lambda + \mathcal{O}(|\lambda|^2). \quad (2.9)$$

In exactly the same way, we find that $\dot{y}_1 = 0$ is solved for by

$$y_1(\lambda) = \frac{-(b+c)(D_0 + D_1 + D_2) - d}{a}\lambda + \mathcal{O}(|\lambda|^2). \quad (2.10)$$

Note that these expressions for $y_0(\lambda)$ and $y_1(\lambda)$ agree up to first order in λ . However, unlike for the synchronous branch, there is no reason to conclude that



(a) The difference between the y -nodes along the steady state branches.

(b) A log-log plot of the difference between the y -nodes, for $\lambda > 0$.

Figure 3: More details for the steady state branches depicted in Figure 1. The black line segment in the log-log plot has slope 3 and was added to show that $y_0(\lambda) - y_1(\lambda) \sim \lambda^3$. Figures taken from [10].

$y_0(\lambda) = y_1(\lambda)$ along this branch, as the equations for \dot{y}_0 and \dot{y}_1 in (2.3) are different for distinct x_0, x_1 and x_2 :

$$\begin{aligned} F(y_0, (x_0(\lambda), x_1(\lambda)), (x_1(\lambda), x_2(\lambda)), (x_2(\lambda), x_0(\lambda)); \lambda) &= 0, \\ F(y_1, (x_0(\lambda), x_2(\lambda)), (x_1(\lambda), x_0(\lambda)), (x_2(\lambda), x_1(\lambda)); \lambda) &= 0. \end{aligned}$$

We will refer to the branch of steady states given by Equations (2.7), (2.9) and (2.10) as the *reluctant branch* of (2.3).

Figure 1 demonstrates numerically that the reluctant branch is truly non-synchronous. The figure was taken from [10], and it shows a numerically obtained plot of the asymptotically stable bifurcation branches that emerge in a bifurcation in a particular realisation of system (2.3), namely for the choices

$$\begin{aligned} G(X_0, X_1, X_2, \lambda) &= -X_0 + X_1 - X_2 + 8\lambda X_0 + 4X_0^2 \quad \text{and} \\ F(Y, (X_0, X_1), (X_2, X_3), (X_4, X_5), \lambda) &= -5Y + 14\lambda \\ &\quad - h(10X_0 - 12X_1) - h(10X_2 - 12X_3) - h(10X_4 - 12X_5) \end{aligned}$$

in which

$$h(x) = \sin(x) + \cos(x) - 1. \quad (2.11)$$

Figure 1 shows the synchronous branch for $\lambda < 0$ and the reluctant branch for $\lambda > 0$ – indeed, y_0 and y_1 agree for $\lambda < 0$, and quite clearly do not for $\lambda > 0$. This becomes even more visible in Figure 3a, which shows $y_0 - y_1$ as a function of λ . The logarithmic plot in Figure 3b suggests that $y_0(\lambda) - y_1(\lambda) \sim \lambda^3$. In Section 4, we prove that this is truly the case.

In this article, we rigorously prove the existence of reluctant steady state branches in bifurcations in a large class of hypernetwork systems. Specifically, we will show that reluctant bifurcation branches appear in generic one-parameter synchrony breaking steady state bifurcations in the admissible ODEs for such hypernetworks. We also provide a formula for the order in λ with which the “reluctant nodes” separate.

3 Preliminaries

We now briefly introduce our formalism for hypernetwork dynamical systems. See [10] for a more detailed exposition. In fact, in the latter paper, we define a hypernetwork to be a collection $\mathbf{N} = (V, H, s, t)$ consisting of a finite set of *vertices* or *nodes* V , a finite set of *hyperedges* H , and source and target maps s and t defined on H . Given an edge $h \in H$, its target $t(h) \in V$ is a single vertex, whereas the source $s(h) = (s_1(h), \dots, s_{k_h}(h)) \in V^{k_h}$ is an ordered k_h -tuple of vertices. The number $k_h > 0$ depends on the hyperedge h , and is called its *order*. To avoid cluttered notation though, we often suppress the dependence of k_h on h when it is clear from context, and simply write k . Note that the k_h vertices in $s(h)$ are not required to be distinct. The order of a hypernetwork \mathbf{N} is then defined as the maximum of the orders of its hyperedges, so that the hypernetworks of order 1 are precisely the classical (dyadic) networks.

In addition to the data that is explicitly given in $\mathbf{N} = (V, H, s, t)$, we also specify equivalence relations on both the nodes V and the hyperedges H . We typically refer to both as the *color* or *type* relation. The reason that these relations are not specified in \mathbf{N} is because they will apply to all hypernetworks at once. That is, it will make sense for two nodes in different hypernetworks to have the same color, and likewise for multiple hyperedges across different hypernetworks. This allows us to define node- and hyperedge-type preserving maps between different hypernetworks, which in turn give rise to semi-conjugacies between the dynamics, see [10] for more details. Intuitively, this color-relation conveys whether two nodes correspond to comparable or incomparable agents in a real-world system modelled by the hypernetwork, and similarly whether or not two hyperedges specify the same influence. As is suggested by this interpretation, the vertex- and hyperedge-types have to satisfy certain consistency conditions. These are:

1. if two nodes v_0 and v_1 are of the same type, then there exists a hyperedge-type preserving bijection between the set of hyperedges targeting v_0 and those targeting v_1 ;
2. two hyperedges h_0, h_1 of the same type have the same order k , and for each $i \in \{1, \dots, k\}$ the nodes $s_i(h_0)$ and $s_i(h_1)$ are of the same vertex-type.

As mentioned, the conditions above can be seen as natural for real-world coupled systems. Their motivation also comes from the fact that they allow us to define dynamical systems that reflect the given hypernetwork structure.

These systems are specified by so-called *admissible* vector fields. To introduce these, we fix an internal phase space \mathbb{R}^{n_v} for each node $v \in V$, which will be identical for nodes of the same type. Each node v is given a state variable $x_v \in \mathbb{R}^{n_v}$. The *total phase space* of the hypernetwork dynamical system describes the states of all these variables, and is thus the direct sum $\bigoplus_{v \in V} \mathbb{R}^{n_v}$. We also specify, for each hyperedge $h \in H$, the vector of its source variables

$$\mathbf{x}_{s(h)} = \left(x_{s_1(h)}, \dots, x_{s_{k_h}(h)} \right) \in \bigoplus_{i=1}^{k_h} \mathbb{R}^{n_{s_i(h)}}.$$

Next, we choose for each node $v \in V$ its *response function*

$$F_v : \bigoplus_{h:t(h)=v} \bigoplus_{i=1}^{k_h} \mathbb{R}^{n_{s_i(h)}} \rightarrow \mathbb{R}^{n_v}. \quad (3.1)$$

These functions must satisfy certain conditions that reflect our intuitive idea that hyperedges of the same type encode identical influence, as well as the notion that nodes of the same type respond to their input in the same way. In words, we require that the variables of identical-type hyperedges may be freely interchanged in F_v , as well as that F_v and F_w are the same when the nodes v and w are of the same type, after an appropriate identification of their domains. We may capture both requirements in one succinct condition as follows: given nodes v and w of the same type, for any hyperedge-type preserving bijection $\alpha : t^{-1}(v) \rightarrow t^{-1}(w)$ we have

$$F_w \left(\bigoplus_{t(h_2)=w} \mathbf{x}_{s(h_2)} \right) = F_v \left(\bigoplus_{t(h_1)=v} \mathbf{x}_{s(\alpha(h_1))} \right), \quad (3.2)$$

for all $x = \bigoplus_{v \in V} x_v \in \bigoplus_{v \in V} \mathbb{R}^{n_v}$. Recall that at least one such α exists when v and w are of the same type. Finally, we define the *hypernetwork admissible vector field*

$$f^{\mathbf{N}} : \bigoplus_{v \in V} \mathbb{R}^{n_v} \rightarrow \bigoplus_{v \in V} \mathbb{R}^{n_v},$$

on the total phase space, to be given component-wise by

$$f_v^{\mathbf{N}}(x) = F_v \left(\bigoplus_{h:t(h)=v} \mathbf{x}_{s(h)} \right)$$

for all $v \in V$ and $x \in \bigoplus_{v \in V} \mathbb{R}^{n_v}$.

Of particular interest in this paper are so-called *augmented hypernetworks*, also introduced in [10]. Their definition involves the symmetric group on $k+1$ elements, denoted by S_{k+1} , which acts on the ordered set $\{0, \dots, k\}$ by permutations. We denote by S_{k+1}^0 and S_{k+1}^1 the subsets of even and odd permutations, respectively, and denote by $\text{sgn}(\sigma) \in \{0, 1\}$ the sign of a permutation $\sigma \in S_{k+1}^{\text{sgn}(\sigma)}$.

Definition 3.1 (Definition 5.1 in [10]). Let \mathbf{N} be a hypernetwork with $k+1 \geq 3$ nodes v_0, \dots, v_k of the same type. We define the *augmented hypernetwork with core \mathbf{N}* , denoted by \mathbf{N}^\diamond , as the hypernetwork obtained by adding two additional nodes w_0, w_1 , one self-loop for each w_i and $(k+1)!$ new hyperedges to \mathbf{N} . The new nodes are of the same type, which differs from that of the v_i . Likewise, we construct a new hyperedge-type which we assign to the $(k+1)!$ additional hyperedges. Necessarily the self-loops on the new nodes are of a same, new type

too. The $(k+1)!$ new hyperedges are indexed by the symmetric group S_{k+1} , so that we may denote them by h_σ for $\sigma \in S_{k+1}$. We define their sources and targets by

$$\begin{aligned} t(h_\sigma) &= w_{\text{sgn}(\sigma)} \text{ and} \\ s(h_\sigma) &= (v_{\sigma(1)}, \dots, v_{\sigma(k)}), \end{aligned} \tag{3.3}$$

where S_{k+1} acts on the ordered set $\{0, \dots, k\}$. Note that $v_{\sigma(0)} \in \{v_0, \dots, v_k\}$ is therefore the only v -node not in the source of h_σ , and that these hyperedges all have order k .

Example 3.2. The hypernetwork discussed in Section 2 and depicted in Figure 2, is an example of an augmented hypernetwork. Here the core consists of the $k+1 = 3$ circular nodes in the center and the arrows between them. This core in fact forms a classic (dyadic) network. The two “added” nodes are depicted as the square ones. Equation (2.1) gives the form of a general admissible vector field for this augmented hypernetwork. Recall that the response function F in Equation (2.1) is invariant under permutations of the three pairs of inputs, which reflects that the six “added” hyperedges are all of the same type.

By assumption, all nodes in the core \mathbf{N} of an augmented hypernetwork are of the same type, so that \mathbf{N}^\diamond has precisely two node-types. This means that two response functions are required to describe an admissible vector field $f^{\mathbf{N}^\diamond}$ for \mathbf{N}^\diamond . We will usually denote these by F and G , where G is used for the nodes in \mathbf{N} and F for the two additional nodes. Likewise, we see that the total phase space is determined by two vector spaces: one for the internal dynamics of the v -nodes, \mathbb{R}^{n_v} , and one for that of the w -nodes, \mathbb{R}^{n_w} . We will later set both equal to \mathbb{R} . Note that F takes one argument from \mathbb{R}^{n_w} , corresponding to the self-loop, and $(k+1)!/2$ entries from $\bigoplus^k \mathbb{R}^{n_v}$ for the remaining hyperedges. As these latter hyperedges are indexed by (half of) the symmetric group, we may see the response function as

$$F: \mathbb{R}^{n_w} \oplus \bigoplus_{\sigma \in S_{k+1}^0} \left(\bigoplus^k \mathbb{R}^{n_v} \right) \rightarrow \mathbb{R}^{n_w},$$

with the property that the $(k+1)!/2$ entries with values in $\bigoplus^k \mathbb{R}^{n_v}$ may be freely interchanged. Note that we simply index these entries by S_{k+1}^0 to emphasize that they correspond to the hyperedges that are indexed by (part of) the symmetric group. We could have also used S_{k+1}^1 and, because we may freely interchange these entries, we do not have to give an explicit identification between S_{k+1}^0 and S_{k+1}^1 . In particular, in any augmented hypernetwork the dynamics of the w -nodes may simply be written as

$$\dot{y}_0 = F \left(y_0, \bigoplus_{\sigma \in S_{k+1}^0} \mathbf{x}_\sigma \right) \text{ and } \dot{y}_1 = F \left(y_1, \bigoplus_{\sigma \in S_{k+1}^1} \mathbf{x}_\sigma \right), \tag{3.4}$$

where $\mathbf{x}_\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(k)})$, and where we write y_i for the state of node w_i and x_j for the state of node v_j .

Remark 1. We may generalise the definition of an augmented hypernetwork by connecting the auxiliary nodes w_0 and w_1 only to a subset \mathcal{S} of nodes of the same type of the core, consisting of $k + 1$ nodes of the same type. In this case, the nodes in the core that are not elements of \mathcal{S} may in fact be of different type. We can then add $(k + 1)!$ hyperedges of order k , precisely as in Definition 3.1, but now with all source nodes in \mathcal{S} . It will be clear that our results also hold for such hypernetworks, but to avoid a cluttered exposition we will mostly work with Definition 3.1. See also Remark 4.

4 Reluctant synchrony-breaking

We now show that the reluctant synchrony breaking observed in Section 2 is not a peculiarity of systems of the form (2.3), but can occur in any augmented hypernetwork. In fact, we shall give natural conditions on the core \mathbf{N} that guarantee that reluctant synchrony breaking occurs generically in the augmented hypernetwork \mathbf{N}^\diamond . Moreover, in Theorem 4.2 below we give a precise expression for the degree (in the bifurcation parameter) at which the reluctant synchrony breaking occurs. We start by introducing some useful notation and conventions.

As it is sometimes convenient to make explicit the dependence of an admissible vector field on its response functions, we will often write $f_{(F,G)}^{\mathbf{N}^\diamond}$ and $f_{(G)}^{\mathbf{N}}$ for the admissible vector fields of \mathbf{N}^\diamond and \mathbf{N} , respectively. Furthermore, because in this section we are mainly interested in bifurcations, we will often use $f^{\mathbf{N}}$ (and $f^{\mathbf{N}^\diamond}$, $f_{(F,G)}^{\mathbf{N}^\diamond}$ etc.) to denote parameter families of admissible vector fields. This means $f^{\mathbf{N}}$ is an admissible vector field for any fixed value of the bifurcation parameter, as in Section 2.

Throughout this section we will investigate asymptotics and power series in λ for bifurcation branches. Some of these might involve fractional powers of λ , meaning that such branches are only defined for positive or negative values of λ . To keep this section as readable as possible, we assume from here on out that all branches are defined for positive values of λ , so that we may always write λ^p for any power $p \geq 0$. The corresponding results for negative values of λ follow easily by redefining λ as $-\lambda$.

Definition 4.1. Let \mathbf{N} be a hypernetwork with n nodes and denote by $f^{\mathbf{N}}: \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ a one-parameter family of admissible vector fields for \mathbf{N} , where each node has a one-dimensional internal phase space. In this paper, a locally defined branch of steady states $x(\lambda) = (x_1(\lambda), \dots, x_n(\lambda))$ for $f^{\mathbf{N}}$ is called *fully synchrony-breaking* if for all $i, j \in \{1, \dots, n\}$ with $i > j$ there exist numbers $p_{i,j} > 0$ and $D_{i,j} \neq 0$ such that

$$x_i(\lambda) - x_j(\lambda) = D_{i,j} \lambda^{p_{i,j}} + \text{“higher order terms” in } \lambda. \quad (4.1)$$

Given a fully synchrony-breaking steady state branch, we define its *order of*

asynchrony as the number

$$\bar{p} = \sum_{\substack{i,j=1 \\ i>j}}^n p_{i,j}. \quad (4.2)$$

In what follows, we will need to make some assumptions about the asymptotics and regularity of the branches. To avoid a detailed exposition, we instead use a formal ansatz. From here on out, we will always assume a fully synchrony-breaking branch $x(\lambda) = (x_1(\lambda), \dots, x_n(\lambda))$ to come with a finite set of numbers $\Upsilon \subseteq (0, \bar{p}]$ such that we may write

$$x_i(\lambda) = \sum_{p \in \Upsilon} A_{i,p} \lambda^p + \mathcal{O}(|\lambda|^{\bar{p}+\epsilon}) \quad (4.3)$$

for some $\epsilon > 0$, and with $A_{i,p} \in \mathbb{R}$. Note that, in contrast to the $D_{i,j}$ in Equation (4.1), these $A_{i,p}$ may very well vanish. As we may write

$$\begin{aligned} x_i(\lambda) - x_j(\lambda) &= \sum_{p \in \Upsilon} (A_{i,p} - A_{j,p}) \lambda^p + \mathcal{O}(|\lambda|^{\bar{p}+\epsilon}) \\ &= D_{i,j} \lambda^{p_{i,j}} + \text{“higher order terms”}, \end{aligned} \quad (4.4)$$

and because $p_{i,j} \leq \bar{p}$, we see that necessarily $p_{i,j} \in \Upsilon$ for all $i > j$.

In the theorem below, we assume all admissible vector fields correspond to one-dimensional internal dynamics for each node.

Theorem 4.2. *Let \mathbf{N}^\diamond be an augmented hypernetwork with core \mathbf{N} , the latter consisting of $k+1 \geq 3$ nodes, and let $f_{(G)}^{\mathbf{N}} : \mathbb{R}^{k+1} \times \Omega \rightarrow \mathbb{R}^{k+1}$ be a one-parameter family of admissible vector fields for \mathbf{N} , corresponding to some response function G . Assume $f_{(G)}^{\mathbf{N}}$ admits a fully synchrony-breaking branch of steady states $x(\lambda) = (x_0(\lambda), \dots, x_k(\lambda))$ with order of asynchrony \bar{p} .*

Then for a generic λ -dependent response function F , the system $f_{(F,G)}^{\mathbf{N}^\diamond} : \mathbb{R}^{k+3} \times \Omega \rightarrow \mathbb{R}^{k+3}$ admits a steady state branch

$$z(\lambda) = (x_0(\lambda), \dots, x_k(\lambda), y_0(\lambda), y_1(\lambda))$$

for which

$$y_0(\lambda) - y_1(\lambda) = E \lambda^{\bar{p}} + \mathcal{O}(|\lambda|^{\bar{p}+\epsilon}),$$

for some non-zero $E \in \mathbb{R}$.

Moreover, $z(\lambda)$ is a branch of steady states for $f_{(-F,G)}^{\mathbf{N}^\diamond}$ as well, and if $x(\lambda)$ is stable for $f_{(G)}^{\mathbf{N}}$, then $z(\lambda)$ is stable for either $f_{(F,G)}^{\mathbf{N}^\diamond}$ or $f_{(-F,G)}^{\mathbf{N}^\diamond}$.

Before proving the theorem, we first apply it to our running example.

Example 4.3. In Section 2, we investigated a bifurcation scenario with a fully synchrony-breaking branch in the core, given by $x_i(\lambda) = D_i \lambda + \mathcal{O}(|\lambda|^2)$ for $i \in \{0, 1, 2\}$ and with mutually distinct D_i . It follows that $x_i(\lambda) - x_j(\lambda) =$

$(D_i - D_j)\lambda + \mathcal{O}(|\lambda|^2)$ for all i, j . As $D_i - D_j \neq 0$ for $i > j$, we obtain $p_{3,1} = p_{3,2} = p_{2,1} = 1$ and hence

$$\bar{p} = 1 + 1 + 1 = 3.$$

Theorem 4.2 therefore predicts a bifurcation branch

$$z(\lambda) = (x_0(\lambda), x_1(\lambda), x_2(\lambda), y_0(\lambda), y_1(\lambda))$$

in the augmented system $f_{F,G}^{\mathbf{N}^\diamond}$, for generic choice of F and with any G supporting the aforementioned fully synchrony-breaking branch in the core, which satisfies

$$y_0(\lambda) - y_1(\lambda) \sim \lambda^3.$$

This is indeed what we found in our numerical investigation, see Figure 3b.

The proof of Theorem 4.2 requires some machinery from [10]. There we introduced the polynomials $P_{(k)}$, given by

$$\begin{aligned} P_{(k)}: \bigoplus_{\sigma \in S_{k+1}^0} \mathbb{R}^k &\rightarrow \mathbb{R} \\ P_{(k)} \left(\bigoplus_{\sigma \in S_{k+1}^0} \mathbf{x}_\sigma \right) &= \sum_{\sigma \in S_{k+1}^0} X_{\sigma,1}^1 X_{\sigma,2}^2 \cdots X_{\sigma,k}^k, \end{aligned} \quad (4.5)$$

for $k \in \mathbb{N}$, and where $\mathbf{x}_\sigma = (X_{\sigma,1}, \dots, X_{\sigma,k}) \in \mathbb{R}^k$ for $\sigma \in S_{k+1}^0$. We also state the following result, a proof of which can be found in [10].

Lemma 4.4 (Lemma 5.6 in [10]). *Let*

$$Q: \bigoplus_{\sigma \in S_{k+1}^0} \mathbb{R}^k \rightarrow \mathbb{R}$$

be a polynomial function that is invariant under all permutations of its $\#S_{k+1}^0$ entries from \mathbb{R}^k . Then there exists a polynomial $S: \mathbb{R}^{k+1} \rightarrow \mathbb{R}$ such that

$$Q \left(\bigoplus_{\sigma \in S_{k+1}^0} \mathbf{x}_\sigma \right) - Q \left(\bigoplus_{\sigma \in S_{k+1}^1} \mathbf{x}_\sigma \right) = S(x) \prod_{\substack{i,j=0 \\ i>j}}^k (x_i - x_j) \quad (4.6)$$

for all $x = (x_0, \dots, x_k) \in \mathbb{R}^{k+1}$, where $\mathbf{x}_\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(k)})$ for all $\sigma \in S_{k+1}$.

Remark 2. It can readily be seen that for any polynomial Q satisfying the conditions of Lemma 4.4, (4.6) is actually equivalent to the fact that

$$Q \left(\bigoplus_{\sigma \in S_{k+1}^0} \mathbf{x}_\sigma \right) = Q \left(\bigoplus_{\sigma \in S_{k+1}^1} \mathbf{x}_\sigma \right) \quad (4.7)$$

whenever $(x_0, \dots, x_k) \in \mathbb{R}^{k+1}$ satisfies $x_i = x_j$ for some distinct $i, j \in \{0, \dots, k\}$. This observation still holds when Q is not polynomial, see Lemma 5.5 of [10]. The latter fact actually underlies the proof of Lemma 4.4 that is given in [10].

Lemma 4.5. *The polynomials $P_{(k)}$ defined in (4.5) satisfy*

$$P_{(k)} \left(\bigoplus_{\sigma \in S_{k+1}^0} \mathbf{x}_\sigma \right) - P_{(k)} \left(\bigoplus_{\sigma \in S_{k+1}^1} \mathbf{x}_\sigma \right) = \prod_{\substack{i,j=0 \\ i>j}}^k (x_i - x_j) \quad (4.8)$$

for all $x = (x_0, \dots, x_k) \in \mathbb{R}^{k+1}$.

Proof. By Lemma 4.4 we have

$$P_{(k)} \left(\bigoplus_{\sigma \in S_{k+1}^0} \mathbf{x}_\sigma \right) - P_{(k)} \left(\bigoplus_{\sigma \in S_{k+1}^1} \mathbf{x}_\sigma \right) = S(x) \prod_{\substack{i,j=0 \\ i>j}}^k (x_i - x_j)$$

for some polynomial S . It remains to show that $S = 1$. To this end, note that both the left and right hand side of Equation (4.8) has total degree $1 + \dots + k = \frac{k(k+1)}{2}$. This means S is a constant polynomial. As both sides of Equation (4.8) contain a term $1 \cdot x_1 x_2^2 \dots x_k^k$, we see that $S = 1$ and the result follows. \square

Before we move on to the proof of Theorem 4.2, we first have a closer look at the set of powers Υ . Recall that we may write

$$x_i(\lambda) = \sum_{p \in \Upsilon} A_{i,p} \lambda^p + \mathcal{O}(|\lambda|^{\bar{p}+\epsilon}) \quad (4.9)$$

for all the components $x_i(\lambda)$ of a fully synchrony-breaking branch. By adding zero-coefficients $A_{i,p}$ to Expression (4.9) and by decreasing ϵ if needed, we may assume that for all $p, q \in \Upsilon$, we have

$$\begin{aligned} p+q &\in \Upsilon && \text{if } p+q \leq \bar{p}; \\ p+q &\geq \bar{p} + \epsilon && \text{if } p+q > \bar{p}, \end{aligned} \quad (4.10)$$

and also that

$$\begin{aligned} 1 &\in \Upsilon && \text{if } 1 \leq \bar{p}; \\ 1 &\geq \bar{p} + \epsilon && \text{if } 1 > \bar{p}. \end{aligned}$$

More precisely, we can add to Υ all non-zero sums $s = c + \sum_{p \in \Upsilon} c_p p$ with non-negative integer coefficients c, c_p , such that $s \leq \bar{p}$. Note that this adds a finite number of elements to Υ , as necessarily $c_p \leq \left\lceil \frac{\bar{p}}{p} \right\rceil$ and $c \leq \lceil \bar{p} \rceil$. It then follows from Equation (4.10) that $\bar{p} \in \Upsilon$, as we have $p_{i,j} \in \Upsilon$ for all $i > j$. This allows us to iteratively investigate coefficients corresponding to (possibly non-integer) powers of λ in the branches, as well as in polynomial expressions involving the components of these branches. For instance, if $x_1(\lambda)$ and $x_2(\lambda)$ are given by Equation (4.9), then we may likewise write

$$x_1(\lambda)x_2(\lambda) = \sum_{p \in \Upsilon} A'_p \lambda^p + \mathcal{O}(|\lambda|^{\bar{p}+\epsilon}) \text{ and}$$

$$\lambda x_1(\lambda) = \sum_{p \in \Upsilon} A_p'' \lambda^p + \mathcal{O}(|\lambda|^{\bar{p}+\epsilon})$$

for some $A_p', A_p'' \in \mathbb{R}$. Finally, whenever

$$w(\lambda) = \sum_{p \in \Upsilon} A_p \lambda^p + \mathcal{O}(|\lambda|^{\bar{p}+\epsilon})$$

for some locally defined map $w: \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ and with $A_p \in \mathbb{R}$, then for $q \in \Upsilon$ we may write

$$[w(\lambda)]_{\leq q} := \sum_{\substack{p \in \Upsilon \\ p \leq q}} A_p \lambda^p \quad \text{and} \quad [w(\lambda)]_{< q} := \sum_{\substack{p \in \Upsilon \\ p < q}} A_p \lambda^p$$

for the truncated power series.

Proof of Theorem 4.2. By assumption, $x(\lambda) = (x_0(\lambda), \dots, x_k(\lambda))$ locally solves $(f_{(F,G)}^{\mathbf{N}^\diamond}(x, y; \lambda))_v = (f_{(G)}^{\mathbf{N}}(x; \lambda))_v = 0$ for all nodes v in the core \mathbf{N} and all $y = (y_0, y_1) \in \mathbb{R}^2$. To solve for the y -components, let $K \in \mathbb{N}$ be such that

$$(K+1) \min(p \mid p \in \Upsilon) > \bar{p}. \quad (4.11)$$

We expand a general response function F as

$$F(Y, \mathbf{X}; \lambda) = aY + \sum_{\ell, m=0}^K Q_{\ell, m}(\mathbf{X}) Y^\ell \lambda^m + \mathcal{O}(\|(\mathbf{X}, Y; \lambda)\|^{K+1}) \quad (4.12)$$

for $Y \in \mathbb{R}$, $\lambda \in \mathbb{R}_{\geq 0}$, and where

$$\mathbf{X} := \bigoplus_{\sigma \in S_{k+1}^0} \mathbf{X}_\sigma$$

with $\mathbf{X}_\sigma \in \mathbb{R}^k$. Here each $Q_{\ell, m}$ is a polynomial of degree at most K that is invariant under all permutations of the vectors \mathbf{X}_σ , which follows from the fact that F is invariant under permutations of these vectors. Our assumption (which is necessary for a bifurcation) that $F(0, 0; 0) = 0$ implies that $Q_{0,0}(0) = 0$. Moreover, by setting the number $a \in \mathbb{R}$ equal to the derivative of F at $(0, 0; 0)$ in the Y -direction, we may assume that $Q_{1,0}(0) = 0$.

For $s \in \{0, 1\}$, the equation $\dot{y}_s = 0$ gives

$$ay_s + \sum_{\ell, m=0}^K Q_{\ell, m} \left(\bigoplus_{\sigma \in S_{k+1}^s} \mathbf{x}_\sigma(\lambda) \right) y_s^\ell \lambda^m + \mathcal{O}(\|(x(\lambda), y_s; \lambda)\|^{K+1}) = 0, \quad (4.13)$$

where $\mathbf{x}_\sigma(\lambda) = (x_{\sigma(1)}(\lambda), \dots, x_{\sigma(k)}(\lambda))$ for all $\sigma \in S_{k+1}$. If we assume $a \neq 0$ then by the implicit function theorem, this equation locally has a unique solution $y_s(\lambda)$, which can be written as

$$y_s(\lambda) = \sum_{p \in \Upsilon} B_{s,p} \lambda^p + \mathcal{O}(|\lambda|^{\bar{p}+\epsilon}).$$

The coefficients $B_{s,p} \in \mathbb{R}$ can iteratively be solved for from the equation

$$ay_s(\lambda) + \sum_{\ell,m=0}^K Q_{\ell,m} \left(\bigoplus_{\sigma \in S_{k+1}^s} \mathbf{x}_\sigma(\lambda) \right) y_s^\ell(\lambda) \lambda^m + \mathcal{O}(|\lambda|^{\bar{p}+\epsilon}) = 0, \quad (4.14)$$

which is (4.13) applied to the solution branch $(x_0(\lambda), \dots, x_k(\lambda), y_0(\lambda), y_1(\lambda))$. We now want to show that

$$y_0(\lambda) - y_1(\lambda) = \mathcal{O}(|\lambda|^{\bar{p}}) \quad (4.15)$$

for these unique solutions. We will do so by proving for all $q \in \Upsilon$ with $q < \bar{p}$ that

$$[y_0]_{<q} = [y_1]_{<q} \implies [y_0]_{\leq q} = [y_1]_{\leq q}. \quad (4.16)$$

Note that for $q = \min(p \mid p \in \Upsilon)$ we have $[y_0]_{<q} = [y_1]_{<q} = 0$. Hence, iterated use of Implication (4.16) indeed proves Equation (4.15). To show that the statement in (4.16) holds, we subtract Equation (4.14) for $s = 1$ from the one for $s = 0$, which gives us

$$a(y_0(\lambda) - y_1(\lambda)) + \sum_{\ell,m=0}^K \sum_{s=0}^1 (-1)^s Q_{\ell,m} \left(\bigoplus_{\sigma \in S_{k+1}^s} \mathbf{x}_\sigma(\lambda) \right) y_s^\ell(\lambda) \lambda^m + \mathcal{O}(|\lambda|^{\bar{p}+\epsilon}) = 0. \quad (4.17)$$

Given $q \in \Upsilon$ satisfying $q < \bar{p}$, let q^+ denote the smallest element in Υ such that $q^+ > q$, i.e. q^+ is the ‘‘next power’’ to consider. Note that $q < \bar{p}$ means $q^+ \leq \bar{p}$ exists. It follows that

$$0 = a([y_0(\lambda)]_{\leq q} - [y_1(\lambda)]_{\leq q}) + \sum_{\ell,m=0}^K \sum_{s=0}^1 (-1)^s Q_{\ell,m} \left(\bigoplus_{\sigma \in S_{k+1}^s} \mathbf{x}_\sigma(\lambda) \right) [y_s(\lambda)]_{<q}^\ell \lambda^m + \mathcal{O}(|\lambda|^{q^+}). \quad (4.18)$$

Here we have used that

$$\begin{aligned} & Q_{\ell,m} \left(\bigoplus_{\sigma \in S_{k+1}^s} \mathbf{x}_\sigma(\lambda) \right) [y_s(\lambda)]_{\leq q}^\ell \lambda^m \\ &= Q_{\ell,m} \left(\bigoplus_{\sigma \in S_{k+1}^s} \mathbf{x}_\sigma(\lambda) \right) [y_s(\lambda)]_{<q}^\ell \lambda^m + \mathcal{O}(|\lambda|^{q^+}), \end{aligned} \quad (4.19)$$

which is clear whenever $\ell = 0$, $\ell > 1$ or $m > 0$. For $(\ell, m) = (1, 0)$ it holds because $Q_{1,0}(0) = 0$, so that $Q_{1,0}$ has no constant term and hence

$$Q_{1,0} \left(\bigoplus_{\sigma \in S_{k+1}^s} \mathbf{x}_\sigma(\lambda) \right)$$

is divisible by λ^r for $r = \min(p \mid p \in \Upsilon)$. We now assume $[y_0(\lambda)]_{<q} = [y_1(\lambda)]_{<q}$, so that $[y_s(\lambda)]_{<q} = [y_0(\lambda)]_{<q}$ for both choices of s . Using Lemma 4.4, Equation (4.18) becomes

$$\begin{aligned}
0 &= a([y_0(\lambda)]_{\leq q} - [y_1(\lambda)]_{\leq q}) \\
&+ \sum_{\ell, m=0}^K \sum_{s=0}^1 (-1)^s Q_{\ell, m} \left(\bigoplus_{\sigma \in S_{k+1}^s} \mathbf{x}_\sigma(\lambda) \right) [y_s(\lambda)]_{<q}^\ell \lambda^m + \mathcal{O}(|\lambda|^{q^+}) \\
&= a([y_0(\lambda)]_{\leq q} - [y_1(\lambda)]_{\leq q}) \\
&+ \sum_{\ell, m=0}^K [y_0(\lambda)]_{<q}^\ell \lambda^m \sum_{s=0}^1 (-1)^s Q_{\ell, m} \left(\bigoplus_{\sigma \in S_{k+1}^s} \mathbf{x}_\sigma(\lambda) \right) + \mathcal{O}(|\lambda|^{q^+}) \quad (4.20) \\
&= a([y_0(\lambda)]_{\leq q} - [y_1(\lambda)]_{\leq q}) \\
&+ \sum_{\ell, m=0}^K [y_0(\lambda)]_{<q}^\ell \lambda^m S_{\ell, m}(x(\lambda)) \prod_{\substack{i, j=0 \\ i > j}}^k (x_i(\lambda) - x_j(\lambda)) + \mathcal{O}(|\lambda|^{q^+}),
\end{aligned}$$

for some polynomials $S_{\ell, m}$. As it is clear that

$$\prod_{\substack{i, j=0 \\ i > j}}^k (x_i(\lambda) - x_j(\lambda)) = \mathcal{O}(|\lambda|^{\bar{p}}),$$

Equation (4.20) simplifies to

$$a([y_0(\lambda)]_{\leq q} - [y_1(\lambda)]_{\leq q}) = \mathcal{O}(|\lambda|^{q^+}).$$

Using again the assumption that $a \neq 0$, this indeed gives $[y_0(\lambda)]_{\leq q} = [y_1(\lambda)]_{\leq q}$.

By induction, (4.15) holds true as outlined above. It follows that we may write

$$y_0(\lambda) - y_1(\lambda) = E\lambda^{\bar{p}} + \mathcal{O}(|\lambda|^{\bar{p}+\epsilon}).$$

for some $E = B_{0, \bar{p}} - B_{1, \bar{p}} \in \mathbb{R}$. We next want to show that $E \neq 0$ generically. To this end, recall that $B_{s, \bar{p}}$ can be solved for from Equation (4.14). In fact, for fixed values of $a \neq 0$ and the power series coefficients of each $x_i(\lambda)$, we may express $B_{s, \bar{p}}$ as a polynomial in the coefficients of the various $Q_{\ell, m}$. Therefore, we may likewise express $E = B_{0, \bar{p}} - B_{1, \bar{p}}$ as such a polynomial. Now, any polynomial on a finite dimensional vector space is either identically zero, or vanishes only on the complement of an open dense set. Therefore, the proof is complete if we can give at least one response function F for which $E \neq 0$. To this end, consider

$$F(Y, \mathbf{X}; \lambda) = aY + P_{(k)}(\mathbf{X}). \quad (4.21)$$

Using Lemma 4.5 we get

$$\begin{aligned}
0 &= F \left(y_0(\lambda), \bigoplus_{\sigma \in S_{k+1}^0} \mathbf{x}_\sigma(\lambda), \lambda \right) - F \left(y_1(\lambda), \bigoplus_{\sigma \in S_{k+1}^1} \mathbf{x}_\sigma(\lambda), \lambda \right) \\
&= a(y_0(\lambda) - y_1(\lambda)) + P_{(k)} \left(\bigoplus_{\sigma \in S_{k+1}^0} \mathbf{x}_\sigma(\lambda) \right) - P_{(k)} \left(\bigoplus_{\sigma \in S_{k+1}^1} \mathbf{x}_\sigma(\lambda) \right) \\
&= a(y_0(\lambda) - y_1(\lambda)) + \prod_{\substack{i,j=0 \\ i>j}}^k (x_i(\lambda) - x_j(\lambda)) \\
&= a(y_0(\lambda) - y_1(\lambda)) + \left(\prod_{\substack{i,j=0 \\ i>j}}^k D_{i,j} \right) \lambda^{\bar{p}} + \mathcal{O}(|\lambda|^{\bar{p}+\epsilon}).
\end{aligned} \tag{4.22}$$

Hence, for this particular choice of response function we obtain

$$E = -\frac{1}{a} \prod_{\substack{i,j=0 \\ i>j}}^k D_{i,j} \neq 0,$$

which shows that E is indeed generically non-vanishing.

Finally, as $(f_{(-F,G)}^{\mathbf{N}^\diamond})_w = -(f_{(F,G)}^{\mathbf{N}^\diamond})_w$ for the two nodes w outside the core, we see that $z(\lambda)$ is a branch of steady-states for $f_{(F,G)}^{\mathbf{N}^\diamond}$, if and only if it is for $f_{(-F,G)}^{\mathbf{N}^\diamond}$. Equation (4.13) shows that the branch

$$z(\lambda) = (x_0(\lambda), \dots, x_k(\lambda), y_0(\lambda), y_1(\lambda)) \tag{4.23}$$

is stable if $x(\lambda)$ is for $f_{(G)}^{\mathbf{N}}$, and if in addition $a < 0$. This last condition holds for either $f_{(F,G)}^{\mathbf{N}^\diamond}$ or $f_{(-F,G)}^{\mathbf{N}^\diamond}$, which completes the proof. \square

Remark 3. The condition that $x(\lambda)$ is fully synchrony-breaking is really essential in Theorem 4.2. More precisely, it follows from Remark 2 that if $x_i(\lambda) = x_j(\lambda)$ for some distinct i, j , then the equations $\dot{y}_0 = 0$ and $\dot{y}_1 = 0$ give identical solutions $y_0(\lambda)$ and $y_1(\lambda)$.

Remark 4. Recall from Remark 1 that we may generalize the definition of an augmented hypernetwork to allow only hyperedges between the two additional nodes and a subset \mathcal{S} of nodes of the same type of the core. It is clear that the results of Theorem 4.2 still hold for this construction. More precisely, we then need a steady-state bifurcation branch in the core that has different components for the nodes in \mathcal{S} . We may then define the order of asynchrony as in Definition 4.1, but comparing only states in \mathcal{S} . As in Theorem 4.2 we will then generically have a steady-state bifurcation in a corresponding admissible vector field for the augmented hypernetwork, with the difference between the

w -nodes growing in λ raised to the power of the order of asynchrony. Stability of this branch for the augmented hypernetwork can be guaranteed if the relevant branch for the core is stable.

5 More examples

In this section, we present three more examples that illustrate Theorem 4.2.

Example 5.1. Consider the augmented hypernetwork shown in Figure 4. Its admissible ODEs are given by

$$\begin{aligned}\dot{x}_0 &= G(x_0, x_1; \lambda), \\ \dot{x}_1 &= G(x_1, x_0; \lambda), \\ \dot{x}_2 &= G(x_2, x_2; \lambda), \\ \dot{y}_0 &= F(y_0, (x_0, x_1), (x_1, x_2), (x_2, x_0); \lambda), \\ \dot{y}_1 &= F(y_1, (x_0, x_2), (x_1, x_0), (x_2, x_1); \lambda),\end{aligned}\tag{5.1}$$

where F has the usual symmetry properties and where we assume each node to have a one-dimensional phase space. The grey box in Figure 4 denotes the core of this hypernetwork, which is a disconnected, classical first-order network, and whose dynamics corresponds to that of the x -variables in (5.1). Generically, a one-parameter bifurcation in an admissible system for the core is either given by the product of two saddle-nodes or by a pitchfork bifurcation. Only the latter of these involves a fully synchrony-breaking branch, and so we focus on that case, which we realise by choosing

$$G(X_0, X_1; \lambda) = -X_0 - X_1 + \lambda X_0 - X_0^3.\tag{5.2}$$

It follows that for $\lambda < 0$, the only (stable) steady state branch is given by $x_0(\lambda) = x_1(\lambda) = x_2(\lambda) = 0$. For $\lambda > 0$ we find (apart from some unstable branches) two stable, fully synchrony-breaking branches, given by

$$x_0(\lambda) = -x_1(\lambda) = \pm\lambda^{1/2}, \quad x_2(\lambda) = 0.$$

For each of these two latter branches we have

$$\begin{aligned}x_2(\lambda) - x_0(\lambda) &= \mp\lambda^{1/2}, \quad x_2(\lambda) - x_1(\lambda) = \pm\lambda^{1/2}, \\ x_1(\lambda) - x_0(\lambda) &= \mp 2\lambda^{1/2},\end{aligned}$$

from which we see that

$$\bar{p} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}.$$

Hence, for the choice of response function G given by (5.2), Theorem 4.2 predicts the full system (5.1) to undergo a bifurcation where $y_0(\lambda) - y_1(\lambda) \sim \lambda^{3/2}$ for $\lambda > 0$, for generic choice of F .

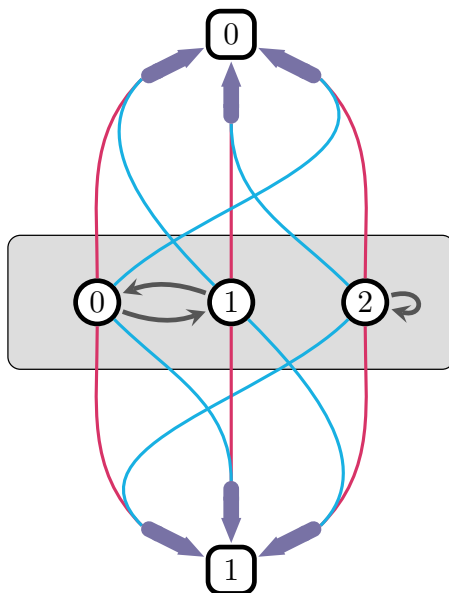


Figure 4: An augmented hypernetwork with a disconnected core, shown within the grey box. We have left out self-loops corresponding to self-influence of each node.

A numerical investigation corroborates this result, see Figure 5. These figures are obtained using Euler’s method for the system (5.1) with G given by Equation (5.2) and where we use

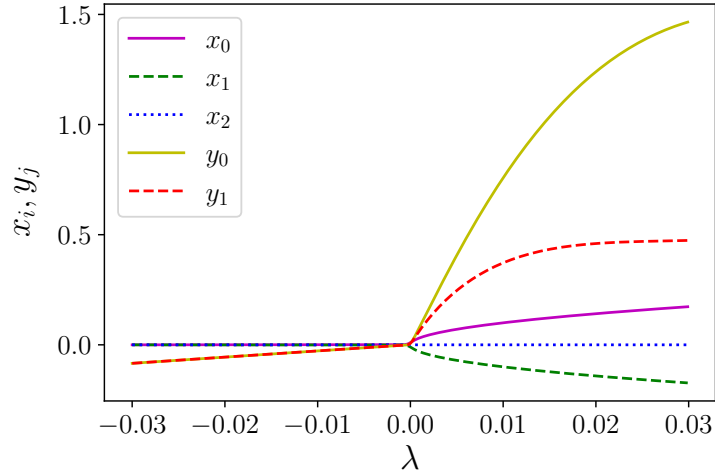
$$\begin{aligned}
 &F(Y, (X_0, X_1), (X_2, X_3), (X_4, X_5); \lambda) \\
 &= -5Y + 14\lambda - h(10X_0 - 12X_1) - h(10X_2 - 12X_3) - h(10X_4 - 12X_5)
 \end{aligned}
 \tag{5.3}$$

in which

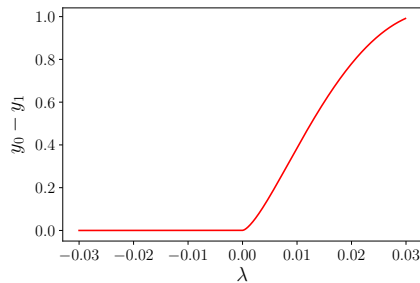
$$h(x) = \sin(x) + \cos(x) - 1 = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right) - 1.
 \tag{5.4}$$

This function F is chosen to facilitate sufficient non-linearity for the result of Theorem 4.2 to hold, while also satisfying the required symmetry condition. We forward integrated the system (5.1) for each of 600 equidistributed values of $\lambda \in [-0.03, 0.03]$. For each fixed value of λ , integration was performed up to $t = 5000$ with time steps of 0.1, and starting from the point $(x_0, x_1, x_2, y_0, y_1) = (0.1, -0.2, 0.3, 0.4, 0.5)$. For the log-log plot of Figure 5c, we instead chose 600 values of $\lambda \in [0.0005, 0.03]$, such that the values of $\ln(\lambda)$ are equidistributed.

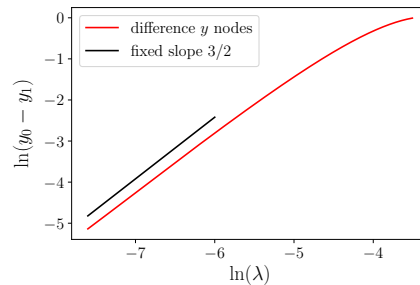
Example 5.2. We next turn to the augmented hypernetwork depicted in Figure 6. The core of this hypernetwork, shown in the grey box, is an example of a classical (dyadic) network that itself shows reluctant synchrony-breaking. More



(a) The stable branches of a synchrony-breaking bifurcation.



(b) The difference between the y -nodes along the stable branches.



(c) A log-log plot of the difference between the y -nodes.

Figure 5: Numerically obtained bifurcation diagram for a system of the form (5.1), corresponding to the augmented hypernetwork shown in Figure 4. The black line segment in the log-log plot has fixed slope $\frac{3}{2}$ for comparison, indicating that $y_0(\lambda) - y_1(\lambda) \sim \lambda^{\frac{3}{2}}$.

precisely, admissible systems for this core are of the form

$$\begin{aligned}\dot{x}_0 &= G(x_0, x_1, x_0; \lambda), \\ \dot{x}_1 &= G(x_1, x_2, x_0; \lambda), \\ \dot{x}_2 &= G(x_2, x_2, x_0; \lambda).\end{aligned}\tag{5.5}$$

These ODEs are special instances of those of the more general form

$$\begin{aligned}\dot{x}_0 &= H(x_0, x_1, x_0, x_1, x_2; \lambda), \\ \dot{x}_1 &= H(x_1, x_2, x_0, x_1, x_2; \lambda), \\ \dot{x}_2 &= H(x_2, x_2, x_0, x_1, x_2; \lambda),\end{aligned}\tag{5.6}$$

obtained by setting $H(x, y, z, u, v; \lambda) = G(x, y, z; \lambda)$. Alternatively, one may think of (5.6) as denoting all admissible systems for a network obtained from the core in Figure 6 by adding six additional arrows. The first three of these are from node 1 to all nodes in the core (including an additional self-loop for node 1), and are all of a single, new type. The last three are from node 2 to all nodes in the core, likewise all of one new type. The reason for adding these new arrow-types is that we can rigorously compute generic steady state bifurcations in systems of the form (5.6), using center manifold reduction. See [24] for a detailed exposition of the techniques used.

For the sake of this example, it is enough to know that in Subsection 7.2 of [24] it is shown that the system (5.6) generically undergoes a steady-state bifurcation involving a synchrony-breaking branch

$$\begin{aligned}x(\lambda) &= (x_0(\lambda), x_1(\lambda), x_2(\lambda)) \\ &= (D_0\lambda + \mathcal{O}(|\lambda|^2), D_1\lambda + \mathcal{O}(|\lambda|^2), D_1\lambda + \mathcal{O}(|\lambda|^2)),\end{aligned}\tag{5.7}$$

where

$$x_2(\lambda) - x_1(\lambda) = D_{2,1}\lambda^2 + \mathcal{O}(|\lambda|^3)\tag{5.8}$$

and with $D_0, D_1, D_0 - D_1, D_{2,1} \neq 0$. As this branch diverges from the synchrony space $\{x_1 = x_2\}$ at only quadratic leading order, we may again speak of reluctant synchrony breaking. As opposed to the reluctant synchrony breaking we have considered in augmented hypernetworks though, the space $\{x_1 = x_2\}$ is actually robust for systems of the form (5.6), and so for the special cases (5.5) as well. The synchrony-breaking branch (5.7) can furthermore take over stability from a fully synchronous one as λ increases through zero, see Table 2.1 in [24]. We therefore predict such a bifurcation to occur in the special system (5.5) as well.

Figure 7 reveals that this is indeed the case. Figure 7a shows the components of the stable branches for the augmented hypernetwork of Figure 6. However, as the core is a subnetwork of its augmented hypernetwork, we see that the x -variables depend only on each other and so depict a bifurcation in the three-node system (5.5) as well. Figure 7a shows a reluctant separation happening between the nodes v_1 and v_2 , corresponding to the variables x_1 and x_2 , with Figures 7b and 7c indicating this occurs as $\sim \lambda^2$.

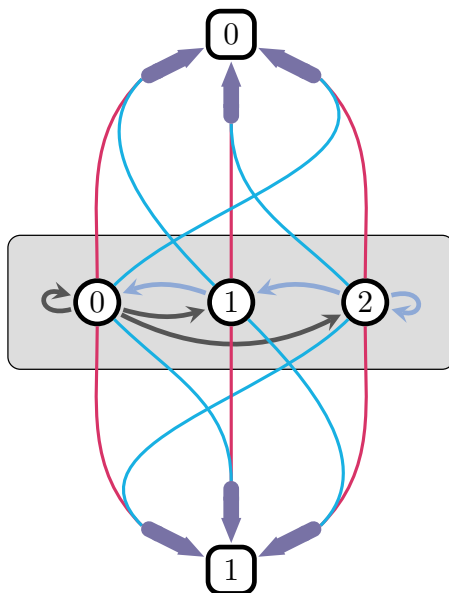


Figure 6: An augmented hypernetwork with a core that itself shows reluctant synchrony breaking. As before, we have left out self-loops corresponding to self-influence of each node.

It follows from (5.7) and (5.8) that we have

$$x_2(\lambda) - x_0(\lambda) \sim \lambda, \quad x_1(\lambda) - x_0(\lambda) \sim \lambda \quad \text{and} \quad x_2(\lambda) - x_1(\lambda) \sim \lambda^2$$

so that

$$\bar{p} = 1 + 1 + 2 = 4.$$

By Theorem 4.2 this implies that the augmented hypernetwork system generically exhibits the highly reluctant synchrony-breaking asymptotics

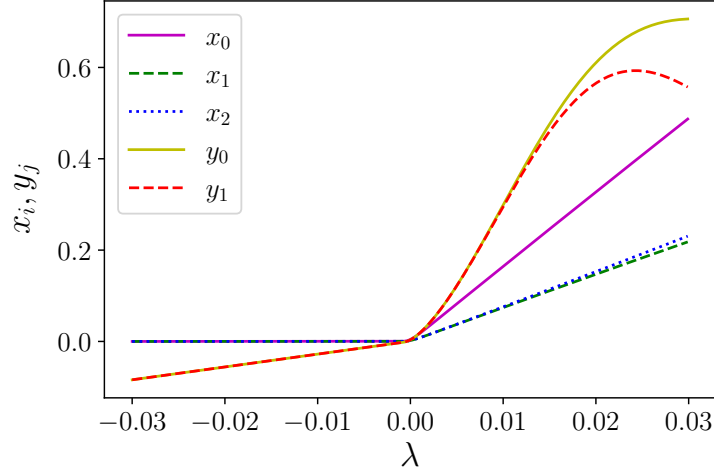
$$y_0(\lambda) - y_1(\lambda) \sim \lambda^4.$$

The numerics in Figures 7d and 7e corroborates this surprising asymptotics – see in particular the branches corresponding to the y -variables in Figure 7a.

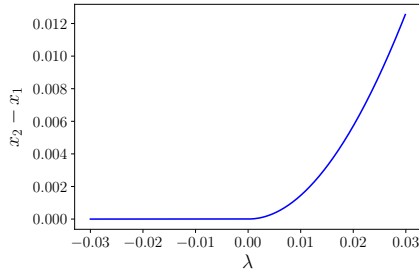
The details of the numerics are the same as for the previous example, except that time ran until $t = 5000$ for Figures 7a, 7b and 7d, and until $t = 15000$ for Figures 7c and 7e. The response function for the x -variables was chosen to be

$$G(X_0, X_1, X_2; \lambda) = -0.55X_1 + 0.25X_2 + 1.5\lambda X_0 - 0.1X_0^2,$$

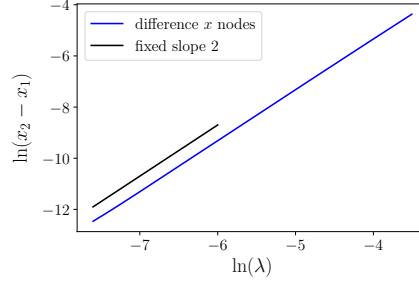
with F given by Equations (5.3) and (5.4).



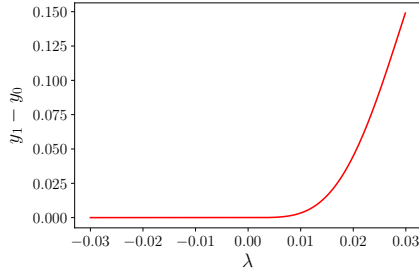
(a) The stable branches of a synchrony-breaking bifurcation.



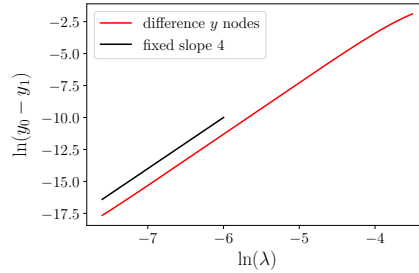
(b) The difference between x_2 and x_1 along the stable branches.



(c) A log-log plot of the difference between x_2 and x_1 .



(d) The difference between the y -nodes along the stable branches.



(e) A log-log plot of the difference between the y -nodes.

Figure 7: Numerically obtained bifurcation diagram for the augmented hypernetwork system shown in Figure 6, whose core is of the form (5.5). The black line segment in (c) has fixed slope 2 for comparison, indicating that $x_2(\lambda) - x_1(\lambda) \sim \lambda^2$. The black line segment in (e) has fixed slope 4, which indicates that $y_0(\lambda) - y_1(\lambda) \sim \lambda^4$.

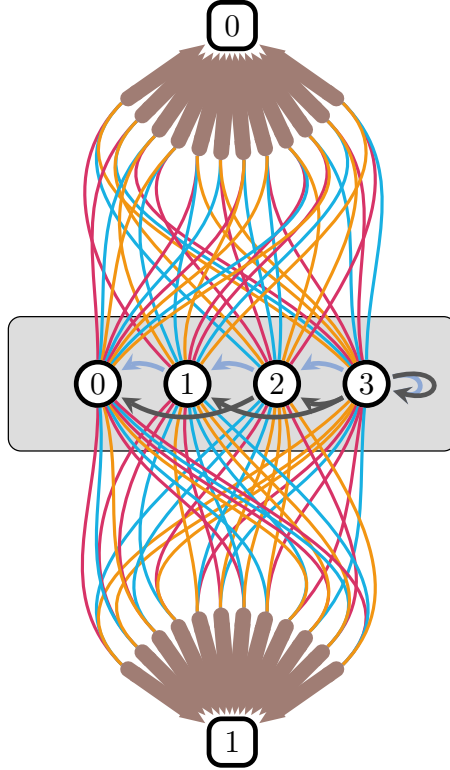


Figure 8: An augmented hypernetwork with a core consisting of a classical feed-forward network with four nodes, shown within the grey box. We have left out self-loops corresponding to self-influence of each node. In contrast to the previous examples, for this hypernetwork we have $k + 1 = 4$, so that there are $(k + 1)! = 24$ hyperedges of order $k = 3$.

Example 5.3. Finally, we consider the augmented hypernetwork of Figure 8, which has as its core a classical feed-forward network with four nodes. More precisely, the nodes in the core evolve according to the ODEs

$$\begin{aligned}
 \dot{x}_0 &= G(x_0, x_1, x_2; \lambda), \\
 \dot{x}_1 &= G(x_1, x_2, x_3; \lambda), \\
 \dot{x}_2 &= G(x_2, x_3, x_3; \lambda), \\
 \dot{x}_3 &= G(x_3, x_3, x_3; \lambda).
 \end{aligned}
 \tag{5.9}$$

It is known that this system generically supports steady-state bifurcations in which stability passes from a fully synchronous branch to one in which

$$x_0(\lambda) \sim \lambda^{1/4}, \quad x_1(\lambda) \sim \lambda^{1/2} \quad \text{and} \quad x_2(\lambda), x_3(\lambda), x_2(\lambda) - x_3(\lambda) \sim \lambda,$$

see [23] and [25]. This unusually fast rate of synchrony breaking is also referred to as *amplification*. It follows that

$$\bar{p} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{2} + \frac{1}{2} + 1 = \frac{11}{4},$$

so that Theorem 4.2 predicts a reluctant steady state branch with

$$y_0(\lambda) - y_1(\lambda) \sim \lambda^{\frac{11}{4}}.$$

This unusual growth rate is verified numerically in Figure 9, which was obtained by numerically integrating the augmented system for

$$G(X_0, X_1, X_2, \lambda) = 10X_1 - 20X_2 + 15\lambda X_0 - 100X_0^2$$

and

$$F\left(Y, \bigoplus_{\sigma \in S_4^0} \mathbf{X}_\sigma; \lambda\right) = -0.01 \sum_{\sigma \in S_4^0} h(120X_{\sigma,1} + 40X_{\sigma,2} - 100X_{\sigma,3}) - 5Y - \lambda,$$

where $\mathbf{X}_\sigma = (X_{\sigma,1}, \dots, X_{\sigma,4}) \in \mathbb{R}^4$ and with h given by Equation (5.4).

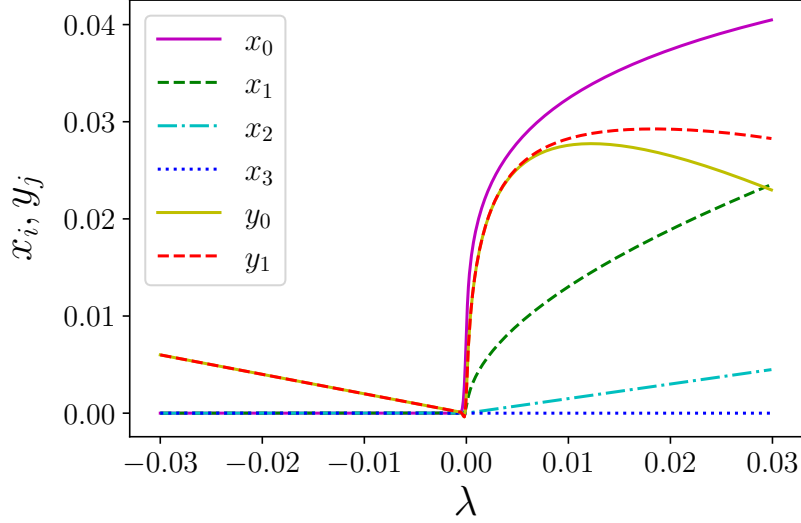
Figure 9a shows the components of the stable branches, whereas Figure 9b is a log-log plot of the difference of the y -components, for positive values of λ . In this latter picture, the dashed black line segment has fixed slope $11/4$, indicating that indeed $y_0(\lambda) - y_1(\lambda) \sim \lambda^{\frac{11}{4}}$. For comparison we also plotted the dotted black line segment, which has fixed slope $10/4$, and which does not fit as well for low values of λ . Both figures were made by forward integrating the vector field for the augmented system from the point

$$(x_0, \dots, x_3, y_0, y_1) = (-0.001, -0.002, -0.003, -0.004, 0.001, 0.002)$$

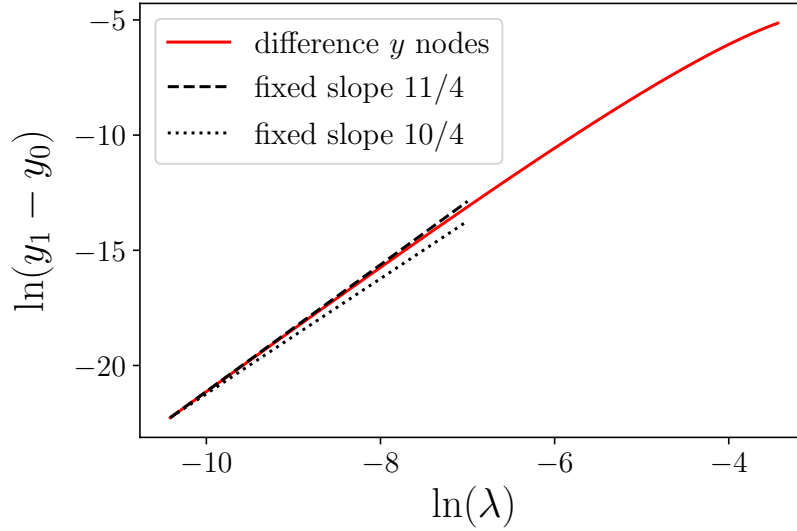
in phase space, for various values of λ and with time steps of 0.1. For Figure 9a this was done up to $t = 2000$ and with 600 equidistant values of $\lambda \in [-0.03, 0.03]$. For Figure 9b this was up to $t = 20000$ and for 100 equidistant values of $\ln(\lambda) \in [\ln(0.00003), \ln(0.03)]$.

Remark 5. Example 5.2 shows that even classical (dyadic) networks may generically support reluctant synchrony breaking bifurcations. However, we are not aware of any method to design networks that, for instance, break synchrony up to some prescribed degree in λ . For hypernetworks, the augmented hypernetwork construction makes this design problem more tractable. In fact, we show below that one may construct hypernetworks which support generic reluctant synchrony breaking to arbitrarily high order. We foresee possible applications of this observation for understanding the phenomenon of homeostasis [26, 27].

To illustrate how one can create hypernetworks with an arbitrarily high order of reluctant synchrony breaking, we return to the five-node augmented hypernetwork of Equation (2.1). In Section 2, we observed that the three-node core of



(a) The stable branches of a synchrony-breaking bifurcation.



(b) A log-log plot of the difference between the y -nodes.

Figure 9: Numerically obtained bifurcation diagram for an augmented hyper-network with core given by a four-node feed-forward network, see Figure 8. The dashed black line segment in the log-log plot has fixed slope $\frac{11}{4}$, whereas the dotted segment has fixed slope $\frac{10}{4}$.

this hypernetwork supports a steady state branch in which $x_i(\lambda) = D_i\lambda + \mathcal{O}(|\lambda|^2)$ for some mutually distinct $D_i \in \mathbb{R}$. Using the same notation as in Section 2, we expand the response function F for the y -nodes as

$$\begin{aligned} & F(Y, (X_0, X_1), (X_2, X_3), (X_4, X_5); \lambda) \\ &= aY + bX_0 + cX_1 + bX_2 + cX_3 + bX_4 + cX_5 + d\lambda \\ &+ \mathcal{O}(\|(Y, X_0, \dots, X_5; \lambda)\|^2), \end{aligned} \quad (5.10)$$

and we recall that we found a reluctant steady state branch in the augmented hypernetwork with asymptotics

$$y_0(\lambda) = \frac{-(b+c)(D_0 + D_1 + D_2) - d}{a}\lambda + \mathcal{O}(|\lambda|^2) \quad (5.11)$$

and

$$y_1(\lambda) = \frac{-(b+c)(D_0 + D_1 + D_2) - d}{a}\lambda + \mathcal{O}(|\lambda|^2). \quad (5.12)$$

To this augmented hypernetwork we can now add another node of the same type as the y -nodes, with corresponding variable y_2 . We couple it to the nodes in the core in such a way that

$$y_2 = F(y_2, (x_0, x_1), (x_1, x_0), (x_0, x_0), \lambda). \quad (5.13)$$

The aforementioned branch of steady states is then supported by this larger hypernetwork as well, where in addition

$$y_2(\lambda) = \frac{-(b+c)(D_0 + D_1 + D_0) - d}{a}\lambda + \mathcal{O}(|\lambda|^2), \quad (5.14)$$

as can be seen using Equation (5.10). To summarise, we now have a branch where the three y -nodes satisfy

$$\begin{aligned} y_0(\lambda) &= E_0\lambda + \mathcal{O}(|\lambda|^2), \quad y_1(\lambda) = E_0\lambda + \mathcal{O}(|\lambda|^2), \\ y_2(\lambda) &= E_2\lambda + \mathcal{O}(|\lambda|^2) \text{ and } y_0(\lambda) - y_1(\lambda) \sim \lambda^p \end{aligned} \quad (5.15)$$

for some $p > 1$ (in this particular case $p = 3$). Moreover, from Equations (5.11), (5.12) and (5.14) we see that $E_0 \neq E_2$, as $D_0 \neq D_2$ by assumption.

We may now use the three y -nodes as the core for another augmented hypernetwork, say by adding two z -nodes of a new type (c.f. Remark 4). We also add a third z -node, precisely as we did with the third y -node. That is, we set

$$\begin{aligned} \dot{z}_0 &= \tilde{F}(z_0, (y_0, y_1), (y_1, y_2), (y_2, y_0), \lambda), \\ \dot{z}_1 &= \tilde{F}(z_1, (y_0, y_2), (y_1, y_0), (y_2, y_1), \lambda), \\ \dot{z}_2 &= \tilde{F}(z_2, (y_0, y_1), (y_1, y_0), (y_0, y_0), \lambda), \end{aligned} \quad (5.16)$$

where \tilde{F} is a response function for the z -nodes. Just as before, we will then find

$$z_0(\lambda) = E'_0\lambda + \mathcal{O}(|\lambda|^2), \quad z_1(\lambda) = E'_0\lambda + \mathcal{O}(|\lambda|^2), \quad (5.17)$$

$$z_2(\lambda) = E'_2\lambda + \mathcal{O}(|\lambda|^2) \text{ and } z_0(\lambda) - z_1(\lambda) \sim \lambda^{p+2}$$

for some generically nonzero $E'_0, E'_2 \in \mathbb{R}$. The new power $p + 2$ follows from Theorem 4.2, as

$$\begin{aligned} y_2(\lambda) - y_0(\lambda) &\sim \lambda, & y_2(\lambda) - y_1(\lambda) &\sim \lambda, \\ y_0(\lambda) - y_1(\lambda) &\sim \lambda^p, \end{aligned} \tag{5.18}$$

which holds because $E_0 \neq E_2$. We may also argue that $E'_0 \neq E'_2$ in precisely the same way that we argued that $E_0 \neq E_2$.

This shows that by iteratively growing the augmented hypernetwork, we may increase the order (in λ) of reluctance of the reluctant steady state branch. In other words, we may design hypernetworks with an arbitrarily high order of reluctant synchrony breaking. Concretely, our example shows that we can arrange for $p = 3, 5, 7, \dots$. It is also clear from this construction that the resulting reluctant branch may be assumed stable.

6 Acknowledgements

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7 Data availability

The code used to generate the numerical results can be found in [28].

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